

Langlands duality for Hitchin systems

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1 Introduction

The purpose of this work is to show that the Hitchin integrable system for a simple complex Lie group G is dual to the Hitchin system for the Langlands dual group ${}^L G$. In particular, the general fiber of the connected component \mathbf{Higgs}_0 of the Hitchin system for G is an abelian variety which is dual to the corresponding fiber of the connected component of the Hitchin system for ${}^L G$. The non-neutral connected components \mathbf{Higgs}_α form torsors over \mathbf{Higgs}_0 . We show that their duals are gerbes over \mathbf{Higgs}_0 which are induced by the gerbe \mathcal{Higgs} of G -Higgs bundles. The latter was introduced and analyzed in [DG02]. More generally, we establish a duality between the gerbe \mathcal{Higgs} of G -Higgs bundles and the gerbe ${}^L \mathcal{Higgs}$ of ${}^L G$ -Higgs bundles, which incorporates all the previous dualities. All these results extend immediately to an arbitrary connected complex reductive group \mathbb{G} .

The Hitchin system $h : \mathbf{Higgs} \rightarrow B$ for the group G and a curve C [Hit87], is an integrable system whose total space is the moduli space of semistable K_C -valued principal G -Higgs bundles on C . The base B parametrizes cameral covers, which are certain

W -Galois covers $p : \tilde{C}_b \rightarrow C$. (W is the Weyl group of G .) For classical groups G , the base B also parametrizes appropriate spectral covers $\bar{p} : \bar{C}_b \rightarrow C$. The Hitchin fiber $h^{-1}(b)$ can be described quite precisely, [Hit87, Fal93, Don93, Don95, Sco98, DG02]. For generic $b \in B$ it is (non-canonically) isomorphic to the product of a finite group and a certain Abelian variety P_b which can be described as a generalized Prym variety of \tilde{C}_b (or of \bar{C}_b) over C .

The connected components of **Higgs** are indexed by the fundamental group $\pi_1(G)$. The component **Higgs**₀ corresponding to the neutral element parametrizes G -Higgs bundles which are induced from G_{sc} -Higgs bundles on C , where G_{sc} is the universal cover group of G . As shown by Hitchin [Hit92] and reviewed here, the restriction of h to this neutral component always admits a section (determined by the choice of a theta characteristic, or spin structure, on the curve C). Our Theorem A shows that the connected components **Higgs**₀ for the groups $G, {}^L G$ are duals of each other.

Here is an outline of the proof of the fiberwise duality, Theorem A. The Hitchin base B , as well as the universal cameral cover $\tilde{\mathcal{C}} \rightarrow C \times B$, depend on the group G only through its Lie algebra \mathfrak{g} . As a first step towards the duality between the Hitchin system $h : \mathbf{Higgs} \rightarrow B$ for G and the Hitchin system ${}^L h : {}^L \mathbf{Higgs} \rightarrow {}^L B$ for ${}^L G$, we note that there is a natural isomorphism $\mathbf{l} : B \rightarrow {}^L B$ between the Hitchin bases for Langlands-dual algebras $\mathfrak{g}, {}^L \mathfrak{g}$. This isomorphism lifts to an isomorphism ℓ of the corresponding universal cameral covers. For the simply laced Lie algebras (of types ADE), this is straightforward: $\mathfrak{g} = {}^L \mathfrak{g}$, $B = {}^L B$, $\tilde{\mathcal{C}} = {}^L \tilde{\mathcal{C}}$, and \mathbf{l}, ℓ are the identity. This is not the case in general. Cameral covers of type B are interchanged with those of type C. For the Lie algebras of types F, G we can identify \mathfrak{g} with ${}^L \mathfrak{g}$ and hence B with ${}^L B$, but the natural isomorphism \mathbf{l} is *not* the identity: it takes one cameral cover to another, in which short and long roots have been interchanged. This phenomenon was recently noted in the Kapustin-Witten work [KW06] on the geometric Langlands Correspondence, and was used in the Argyres-Kapustin-Seiberg work [AKS05] on S -duality in $N = 4$ gauge theories.

Next we show that the group of connected components of a Hitchin fiber $h^{-1}(b)$ is $\pi_1(G)$. In particular, the connected components of $h^{-1}(b)$ are its intersections with the connected components of **Higgs** itself.

For the fiberwise duality, it remains to show that the connected component P_b of the Hitchin fiber $h^{-1}(b)$ over some general $b \in B$ is dual (as a polarized abelian variety) to the connected component ${}^L P_{\mathbf{l}(b)}$ of the corresponding fiber for the Langlands-dual system. This is achieved by analyzing the cohomology of three group schemes $\bar{\mathcal{T}} \supset \mathcal{T} \supset \mathcal{T}^0$ over C attached to a group G . The first two of these were introduced in [DG02], where it was shown that $h^{-1}(b)$ is a torsor over $H^1(C, \mathcal{T})$. We recall the definitions of these two group schemes and add the third, \mathcal{T}^0 , which is simply their connected component. It was noted in [DG02] that $\bar{\mathcal{T}} = \mathcal{T}$ except when $G = \text{SO}(2r + 1)$ for $r \geq 1$. Dually, we note here that $\mathcal{T} = \mathcal{T}^0$ except for $G = \text{Sp}(r)$, $r \geq 1$. In fact, it turns out that the connected components of $H^1(\mathcal{T}^0)$ and $H^1(\bar{\mathcal{T}})$ are dual to the connected components of $H^1({}^L \bar{\mathcal{T}})$, $H^1({}^L \mathcal{T}^0)$, and we are able to identify the intermediate objects $H^1(\mathcal{T}), H^1({}^L \mathcal{T})$ with enough precision to deduce that they are indeed dual to each other.

We extend this result to the non-neutral components as follows. The non-canonical

isomorphism from non-neutral components of the Hitchin fiber to P_b can result in the absence of a section, i.e. in a non-trivial torsor structure [HT03, DP03]. In general, the duality between a family of abelian varieties $A \rightarrow B$ over a base B and its dual family $A^\vee \rightarrow B$ is given by a Poincaré sheaf which induces a Fourier-Mukai equivalence of derived categories. It is well known [DP03, BB06b, BB06a] that the Fourier-Mukai transform of an A -torsor A_α is an \mathcal{O}^* -gerbe ${}_\alpha A^\vee$ on A^\vee . In our case there is indeed a natural stack mapping to **Higgs**, namely the moduli stack **Higgs** of semistable G -Higgs bundles on C . Over the locus of stable bundles, the stabilizers of this stack are isomorphic to the center $Z(G)$ of G and so over the stable locus **Higgs** is a gerbe. The stack **Higgs** was analyzed in [DG02]. From [DP03] we know that every pair $\alpha \in \pi_0(\mathbf{Higgs}) = \pi_1(G), \beta \in \pi_1({}^L G) = Z(G)^\wedge$ defines a $U(1)$ -gerbe ${}_\beta \mathbf{Higgs}_\alpha$ on the connected component \mathbf{Higgs}_α and that there is a Fourier-Mukai equivalence of categories $D^b({}_\beta \mathbf{Higgs}_\alpha) \cong D^b({}_\alpha {}^L \mathbf{Higgs}_\beta)$. In our case we find that all the $U(1)$ -gerbes ${}_\beta \mathbf{Higgs}_\alpha$ are induced from the single $Z(G)$ -gerbe **Higgs**, restricted to component \mathbf{Higgs}_α , via the homomorphisms $\beta : Z(G) \rightarrow U(1)$. We repackage these results in Theorem B, as a duality between the Higgs gerbes **Higgs** and ${}^L \mathbf{Higgs}$. We also note that the gerbe **Higgs** is induced from the gerbe **Bun** on the moduli space **Bun** of stable principal G bundles, which measures the obstruction to existence of a universal bundle. Finally, our Corollary 3.7, allows one to view the Fourier-Mukai duality in Theorem B as a classical limit of the geometric Langlands correspondence: under this duality, the structure sheaves of gerby points on \mathbf{Higgs}_0 are transformed into coherent sheaves on the space ${}^L \mathbf{Higgs}$ (or equivalently Higgs sheaves on ${}^L \mathbf{Bun}$) which are eigensheaves for the Hecke correspondences. The construction of abelianized Hecke eigensheaves, mostly in the case $G = \mathrm{GL}(r)$, was discussed by one of us (R.D.) in several talks in 1990.

Several topological results that are needed in our proof are collected in section 4. The main result of that section is an explicit formula for the cocharacters of the Hitchin Prym.

Several special cases of our result are already known. Hausel and Thaddeus [HT03] considered the case $G = \mathrm{SL}(n)$, ${}^L G = \mathbb{P}\mathrm{GL}(n)$. They also showed the equality of stringy Hodge numbers for these Langlands-dual Hitchin systems and discussed the relationship to mirror symmetry for hyper-Kähler manifolds. The general duality of Hitchin systems is the starting point of Arinkin's approach [Ari02] to the quasi-classical geometric Langlands correspondence. This approach was recently utilized by Bezrukavnikov and Braverman [BB06b] who proved the geometric Langlands correspondence for curves over finite fields and $G = {}^L G = \mathrm{GL}_n$. As explained in [BJSV95], [KW06], the duality of the gerbes **Higgs** and ${}^L \mathbf{Higgs}$ proven in Theorem B was expected to hold on physical grounds.

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2 Duality of Hitchin fibers

In this section we formulate and prove the main duality result for Hitchin Pryms.

2.1 Recollections and notation

2.1.1. Let G be a simple complex algebraic group and let ${}^L G$ be the Langlands dual complex group. The Lie algebras of G and ${}^L G$ will be denoted by \mathfrak{g} and ${}^L \mathfrak{g}$. We fix maximal tori $T \subset G$ and ${}^L T \subset {}^L G$ and denote the corresponding Cartan subalgebras by $\mathfrak{t} \subset \mathfrak{g}$ and ${}^L \mathfrak{t} \subset {}^L \mathfrak{g}$. We will also write $T_{\mathbb{R}} \subset G_{\mathbb{R}}$ and ${}^L T_{\mathbb{R}} \subset {}^L G_{\mathbb{R}}$ for the compact real forms of the complex groups and $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ and ${}^L \mathfrak{t}_{\mathbb{R}} \subset {}^L \mathfrak{g}_{\mathbb{R}}$ will denote the corresponding real Lie algebras. We denote the space of \mathbb{C} -linear functions on \mathfrak{t} by \mathfrak{t}^{\vee} , and the space of \mathbb{R} -linear functions on $\mathfrak{t}_{\mathbb{R}}$ by $\mathfrak{t}_{\mathbb{R}}^{\vee}$. Langlands duality gives an isomorphism $\mathfrak{t}^{\vee} = {}^L \mathfrak{t}$ which is compatible with the real structure. We fix this isomorphism once and for all. We will also write W for the isomorphic Weyl groups of G and ${}^L G$.

We denote the natural pairing between \mathfrak{t} and \mathfrak{t}^{\vee} by $(\bullet, \bullet) : \mathfrak{t}^{\vee} \otimes \mathfrak{t} \rightarrow \mathbb{C}$, while we write $\langle \bullet, \bullet \rangle : \mathfrak{t} \otimes \mathfrak{t} \rightarrow \mathbb{C}$ for a Killing form on \mathfrak{t} . For any group G we have a natural collection of lattices

$$\begin{aligned} \text{root}_{\mathfrak{g}} &\subset \text{char}_G \subset \text{weight}_{\mathfrak{g}} \subset \mathfrak{t}^{\vee} \\ \text{coroot}_{\mathfrak{g}} &\subset \text{cochar}_G \subset \text{coweight}_{\mathfrak{g}} \subset \mathfrak{t}. \end{aligned}$$

Here $\text{root}_{\mathfrak{g}} \subset \text{weight}_{\mathfrak{g}} \subset \mathfrak{t}^{\vee}$ are the root and weight lattice corresponding to the root system on \mathfrak{g} and $\text{char}_G = \text{Hom}(T, \mathbb{C}^{\times}) = \text{Hom}(T_{\mathbb{R}}, S^1)$ is the character lattice of G . Analogously,

$$\begin{aligned} \text{coroot}_{\mathfrak{g}} &= \{x \in \mathfrak{t} \mid (\text{weight}_{\mathfrak{g}}, x) \in \mathbb{Z}\} \cong \text{weight}_{\mathfrak{g}}^{\vee} \\ \text{coweight}_{\mathfrak{g}} &= \{x \in \mathfrak{t} \mid (\text{root}_{\mathfrak{g}}, x) \in \mathbb{Z}\} \cong \text{root}_{\mathfrak{g}}^{\vee} \end{aligned}$$

are the coroot and coweight lattices of G , and

$$\text{cochar}_G = \text{Hom}(\mathbb{C}^{\times}, T) = \text{Hom}(S^1, T_{\mathbb{R}}) = \{x \in \mathfrak{t} \mid (\text{char}_G, x) \in \mathbb{Z}\} \cong \text{char}_G^{\vee}$$

is the cocharacter lattice of G .

The Langlands duality isomorphism ${}^L \mathfrak{t}^{\vee} = \mathfrak{t}$ identifies $\text{root}_{[{}^L \mathfrak{g}]} = \text{coroot}_{\mathfrak{g}}$, $\text{char}_{[{}^L G]} = \text{cochar}_G$, and $\text{weight}_{[{}^L \mathfrak{g}]} = \text{coweight}_{\mathfrak{g}}$. To every root $\alpha \in \text{root}_{\mathfrak{g}}$ of \mathfrak{g} one associates in a standard way a coroot $\alpha^{\vee} \in \text{coroot}_{\mathfrak{g}}$, given by the formula $(\bullet, \alpha^{\vee}) := 2\langle \alpha, \bullet \rangle / \langle \alpha, \alpha \rangle$. Under the identification $\text{root}_{[{}^L \mathfrak{g}]} = \text{coroot}_{\mathfrak{g}}$ the root system of ${}^L \mathfrak{g}$ is mapped to the system of coroots of \mathfrak{g} so that the short and long roots get exchanged.

2.1.2. Let C be a smooth compact complex curve of genus $g > 0$. Let $h : \mathbf{Higgs} \rightarrow B$ and ${}^L h : {}^L \mathbf{Higgs} \rightarrow {}^L B$ denote the Hitchin integrable systems for C and G and ${}^L G$ respectively. Recall [Hit87] that the total space \mathbf{Higgs} parametrizes semistable K_C -valued principal G -Higgs bundles on C . The base B can be identified with the space of sections $H^0(C, (K_C \otimes \mathfrak{t})/W)$. Its points $b \in B$ correspond to certain W -Galois covers $p_b : \tilde{C}_b \rightarrow C$ of C called cameral covers, see [Fal93, Don93, Don95, DG02].

The Hitchin fiber $h^{-1}(b)$ is, in general, disconnected but all of its connected components are torsors over a generalized Prym variety P_b naturally associated with the cover $p_b : \tilde{C}_b \rightarrow C$ [Fal93, Don93, DG02, DDP05]. If G is a classical group the fibers of the Hitchin map can be described [Hit87] as torsors over a generalized Prym variety attached to a (non-Galois) spectral cover $\bar{p}_b : \bar{C}_p \rightarrow C$.

The connected components of the space **Higgs** are labeled by the topological types of Higgs bundles, which in turn are labeled by elements in $H^2(C, \pi_1(G)) = \pi_1(G)$. The component **Higgs**₀ corresponding to the neutral element parametrizes G -Higgs bundles which are induced from G_{sc} -Higgs bundles on C , where G_{sc} is the universal cover group of G . The restriction of h to this neutral component always admits a section (determined by the choice of a theta characteristic on the curve C).

The ramification divisor $D_b \subset \tilde{C}_b$ of a cameral cover $p_b : \tilde{C}_b \rightarrow C$ is a disjoint union $D_b = \coprod_{\alpha} D_b^{\alpha}$ of subdivisors labeled by the roots of \mathfrak{g} [DG02]. We will say that a cameral cover $p_b : \tilde{C}_b \rightarrow C$ has a simple Galois ramification if all ramification points $x \in D_b \subset \tilde{C}_b$ of p have ramification index one. We will denote the universal cameral cover $\tilde{\mathcal{C}} \rightarrow B \times C$. The discriminant $\Delta \subset B$ is the locus of all b for which $p_b : \tilde{C}_b \rightarrow C$ does not have simple Galois ramification.

2.1.3. For a finitely generated abelian group H we will write $H_{\text{tors}} \subset H$ for the torsion subgroup of H ; $H_{\text{tf}} := H/H_{\text{tors}}$ for the maximal torsion free quotient of H ; $H^{\vee} := \text{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ for the dual finitely generated group; $H_{\text{tors}}^{\wedge} := \text{Hom}_{\mathbb{Z}}(H_{\text{tors}}, S^1)$ for the Pontryagin dual of the torsion group H_{tors} .

2.2 Duality for cameral Prym varieties

Theorem A *Let G be a simple complex group, ${}^L G$ the Langlands dual complex group, and C a smooth curve of genus $g > 0$.*

- (1) *There is an isomorphism $\mathbf{l} : B \rightarrow {}^L B$, from the base of the G -Hitchin system to the base of the ${}^L G$ -Hitchin system, which is uniquely determined up to overall scalar, and is such that:*
 - \mathbf{l} *preserves discriminants:* $\mathbf{l}(\Delta) = {}^L \Delta$.
 - \mathbf{l} *lifts to an isomorphism $\ell : \tilde{\mathcal{C}} \rightarrow {}^L \tilde{\mathcal{C}}$ between the universal cameral covers of C .*
- (2) *For $b \in B - \Delta$ there exists a duality of the corresponding G and ${}^L G$ Hitchin fibers, given by an isomorphism of polarized abelian varieties*

$$\mathbf{l}_b : \hat{P}_b \xrightarrow{\cong} {}^L P_{\mathbf{l}(b)},$$

where \hat{P} denotes the dual abelian variety of P . The isomorphism \mathbf{l}_b is the restriction of a global duality of **Higgs**₀ and ${}^L \mathbf{Higgs}_0$ over $B - \Delta$.

Remark 2.1 Given two isomorphic Lie algebras $\mathfrak{g}, \mathfrak{g}'$, there is a canonical isomorphism $W \cong W'$ between their Weyl groups and an isomorphism $\mathfrak{t} \cong \mathfrak{t}'$ between their Cartan subalgebras, taking roots to roots and intertwining the Weyl actions. This isomorphism is unique up to the action of W . For Langlands self-dual algebras, this gives a canonical choice of the Killing form such that the composition $\mathfrak{t} \rightarrow \mathfrak{t}^\vee = {}^L\mathfrak{t} \cong \mathfrak{t}$ sends short roots to long roots. The resulting automorphism of \mathfrak{t} is in W if \mathfrak{g} is simply laced (i.e. of type ADE) but not otherwise (types FG), since it sends long roots to a multiple (greater than 1) of the short roots. The induced automorphism of the base B of the Hitchin system then will be the identity for types ADE but not for types FG. The action of these non-trivial automorphisms of the Hitchin space was recently identified [AKS05] as an S -duality transformation in $N = 4$ gauge theories compatible with a T -duality transformation upon embedding in string theory.

Proof of Theorem A. (1) Recall that $B = H^0(C, (K_C \otimes \mathfrak{t})/W)$ and similarly ${}^LB = H^0(C, (K_C \otimes {}^L\mathfrak{t})/W)$. The choice of a Killing form gives an isomorphism $\kappa : \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^\vee = {}^L\mathfrak{t}$ compatible with the W -action and taking reflection hyperplanes in \mathfrak{t} to reflection hyperplanes in ${}^L\mathfrak{t}$. Therefore the isomorphism $\iota : H^0(C, (K_C \otimes \mathfrak{t})/W) \rightarrow H^0(C, (K_C \otimes {}^L\mathfrak{t})/W)$ induced from κ will preserve discriminants.

The isomorphism κ globalizes to a commutative diagram of bundles over C :

$$\begin{array}{ccc} K_C \otimes \mathfrak{t} & \xrightarrow{\text{id}_{K_C} \otimes \kappa} & K_C \otimes {}^L\mathfrak{t} \\ \downarrow & & \downarrow \\ (K_C \otimes \mathfrak{t})/W & \xrightarrow{\text{id}_{K_C} \otimes \kappa} & (K_C \otimes {}^L\mathfrak{t})/W \end{array}$$

The universal cameral covers $\tilde{\mathcal{C}}$ and ${}^L\tilde{\mathcal{C}}$ are the pullbacks of the columns of this diagram by the natural evaluation maps

$$H^0(C, (K_C \otimes \mathfrak{t})/W) \times C \longrightarrow \text{tot}(K_C \otimes \mathfrak{t})/W$$

$$H^0(C, (K_C \otimes {}^L\mathfrak{t})/W) \times C \longrightarrow \text{tot}(K_C \otimes {}^L\mathfrak{t})/W.$$

The isomorphism $\ell : \tilde{\mathcal{C}} \rightarrow {}^L\tilde{\mathcal{C}}$ is then induced by the isomorphism in the top row of the diagram.

(2) In order to avoid too many exceptional cases in the exposition, we will from now on exclude the case when G is of type A_1 . This case is well understood and recorded in the literature, see e.g. [HT03], [DDD⁺05].

Let $b \in B - \Delta$ and let $\tilde{C} = \mathcal{C}_b$ be the corresponding cameral cover of C , which from now on we identify with the cameral cover ${}^L\tilde{\mathcal{C}}_{\mathcal{U}(b)}$ via the isomorphism ℓ . We will denote the

covering map by $p : \tilde{C} \rightarrow C$. Let $T \subset G$ be the maximal torus of G and let $\Lambda := \text{cochar}_G = \text{Hom}(\mathbb{C}^\times, T)$ be the corresponding cocharacter lattice.

Two sheaves of commutative groups

$$\begin{aligned}\overline{\mathcal{T}} &= p_*(\Lambda \otimes \mathcal{O}_{\tilde{C}}^\times)^W \\ \mathcal{T} &= \left\{ t \in \Gamma(p^{-1}(U), \Lambda \otimes \mathcal{O}_{\tilde{C}}^\times)^W \left| \begin{array}{l} \text{for every root } \alpha \text{ of } \mathfrak{g} \\ \text{we have } \alpha(t)|_{D^\alpha} = 1 \end{array} \right. \right\}\end{aligned}$$

were introduced in [DG02]. In the above formula we identify $\Lambda \otimes \mathbb{C}^\times$ with T and we view a root α as a homomorphism $\alpha : T \rightarrow \mathbb{C}^\times$. The divisor $D^\alpha \subset \tilde{C}$ is the fixed divisor for the reflection $\rho_\alpha \in W$ corresponding to α . To these we now add a third group scheme \mathcal{T}^o , the connected component of \mathcal{T} .

It will be convenient to introduce real forms $\mathcal{T}_{\mathbb{R}}^o$, $\mathcal{T}_{\mathbb{R}}$, and $\overline{\mathcal{T}}_{\mathbb{R}}$ which are defined in the same way but with the holomorphic sheaf $\mathcal{O}_{\tilde{C}}^\times$ replaced by the constant real sheaf S^1 .

By definition we have sheaf inclusions $\mathcal{T}_{\mathbb{R}}^o \subset \mathcal{T}_{\mathbb{R}} \subset \overline{\mathcal{T}}_{\mathbb{R}}$. At any $x \in C$, which is not a branch point of p , the fibers of the three sheaves are equal to each other and non-canonically isomorphic to the compact torus $T_{\mathbb{R}} := \Lambda \otimes S^1$. At a simple branch point $s \in C$ sitting under a ramification point in $D^\alpha \subset \tilde{C}$, the fibers are:

$$(1) \quad \begin{aligned}\overline{\mathcal{T}}_{\mathbb{R},s} &= \{ \lambda \otimes z \mid \alpha^\vee(z^{(\alpha,\lambda)}) = 1 \text{ in } T_{\mathbb{R}} \} \\ \mathcal{T}_{\mathbb{R},s} &= \{ \lambda \otimes z \mid z^{(\alpha,\lambda)} = 1 \text{ in } S^1 \} \\ \mathcal{T}_{\mathbb{R},s}^o &= \{ \lambda \otimes z \mid (\alpha, \lambda) = 0 \text{ in } \mathbb{Z} \}\end{aligned}$$

It was shown in [DG02] that the Hitchin fiber over b is a torsor over the group $H^1(C, \mathcal{T})$. To carry out the comparison with the Hitchin fiber for ${}^L G$, we will also make use of the complex algebraic groups $H^1(C, \mathcal{T}^o)$ and $H^1(C, \overline{\mathcal{T}})$. Our main tool will be the sheaves $\mathcal{T}_{\mathbb{R}}^o$, $\mathcal{T}_{\mathbb{R}}$, and $\overline{\mathcal{T}}_{\mathbb{R}}$. The point is that, as observed in [DDP05], the sheaves \mathcal{T}^o , \mathcal{T} , $\overline{\mathcal{T}}$ have the same first cohomology as their real forms. We briefly recall the argument. The inclusion of groups $S^1 \subset \mathbb{C}^\times$ induces a natural inclusion of sheaves

$$(2) \quad \nu : \Lambda \otimes S^1 \hookrightarrow \Lambda \otimes \mathcal{O}_{\tilde{C}}^\times.$$

We claim that ν induces an isomorphism of commutative Lie groups

$$\begin{array}{ccc} h^1(\nu) : H^1(C, (p_*(\Lambda \otimes S^1))^W) & \longrightarrow & H^1(C, (p_*(\Lambda \otimes \mathcal{O}_{\tilde{C}}^\times))^W) \\ \parallel & & \parallel \\ H^1(C, \overline{\mathcal{T}}_{\mathbb{R}}) & & H^1(C, \overline{\mathcal{T}}). \end{array}$$

Indeed, observe that $H^1(C, (p_*(\Lambda \otimes S^1))^W)$ is isogenous to $H^1(\tilde{C}, \Lambda \otimes S^1)^W$ and similarly $H^1(C, (p_*(\Lambda \otimes \mathcal{O}_{\tilde{C}}^\times))^W)$ is isogenous to $H^1(\tilde{C}, \Lambda \otimes \mathcal{O}_{\tilde{C}}^\times)^W$. Under these isogenies the map $h^1(\nu)$ is compatible with the map

$$H^1(\tilde{C}, \Lambda \otimes S^1)^W \rightarrow H^1(\tilde{C}, \Lambda \otimes \mathcal{O}_{\tilde{C}}^\times)^W$$

and so $h^1(\nu)$ has at most a finite kernel and a discrete cokernel.

Let \mathbf{c} be the cone of the map of sheaves (2). Since the constant sheaf $\mathbb{C}_{\tilde{C}}^\times$ has a resolution

$$\mathbb{C}_{\tilde{C}}^\times \rightarrow \mathcal{O}_{\tilde{C}}^\times \rightarrow \Omega_{\tilde{C}}^1,$$

and since $\mathbb{C}^\times = S^1 \times \mathbb{R}$, it follows that \mathbf{c} is quasi-isomorphic to a complex of \mathbb{R} -vector spaces on \tilde{C} with cohomology sheaves $\mathcal{H}^0 \mathbf{c} \cong \Lambda \otimes \Omega_{\tilde{C}}^1$ (considered as a sheaf of \mathbb{R} -vector spaces), and $\mathcal{H}^1 \mathbf{c} \cong \Lambda \otimes \mathbb{R}$. This implies that

$$\text{cone} \left[(p_*(\Lambda \otimes S^1))^W \rightarrow (p_*(\Lambda \otimes \mathcal{O}_{\tilde{C}}^\times))^W \right]$$

is a complex of sheaves of \mathbb{R} -vector spaces on C and so its hypercohomology can not be a torsion group. This implies that $h^1(\nu)$ is an isomorphism. Next note that by our assumption of simple Galois ramification for $p : \tilde{C} \rightarrow C$ and from the definitions of \mathcal{T} and $\mathcal{T}_{\mathbb{R}}$, it follows that $\overline{\mathcal{T}}/\mathcal{T}$ is a sheaf of groups, which is supported at the branch points of p , and whose stalk at a branch point s is representable by the finite group $\overline{\mathcal{T}}_{\mathbb{R},s}/\mathcal{T}_{\mathbb{R},s}$. Using this fact and the isomorphism $h^1(\nu)$, we can compare the long exact cohomology sequences associated with $0 \rightarrow \mathcal{T} \rightarrow \overline{\mathcal{T}} \rightarrow \overline{\mathcal{T}}/\mathcal{T} \rightarrow 0$ and $0 \rightarrow \mathcal{T}_{\mathbb{R}} \rightarrow \overline{\mathcal{T}}_{\mathbb{R}} \rightarrow \overline{\mathcal{T}}_{\mathbb{R}}/\mathcal{T}_{\mathbb{R}} \rightarrow 0$, to conclude that $H^1(C, \mathcal{T}_{\mathbb{R}}) \cong H^1(C, \mathcal{T})$. The same reasoning also yields the identification $H^1(C, \mathcal{T}_{\mathbb{R}}^\circ) \cong H^1(C, \mathcal{T}^\circ)$.

Recall that a root α for \mathfrak{g} determines a homomorphism $(\alpha, \bullet) : \Lambda_G \rightarrow \mathbb{Z}$. We let $\varepsilon = \varepsilon_{\alpha,G}$ be the positive generator of the image. We also define $\varepsilon^\vee = \varepsilon_{\alpha,G}^\vee := \varepsilon_{\alpha^\vee, L_G}$.

Lemma 2.2 (a) $\varepsilon_{\alpha,G} = 2$ when $G = \text{Sp}(r)$ and α is a long root, and $\varepsilon_{\alpha,G} = 1$ in all other cases. Dually $\varepsilon_{\alpha,G}^\vee = 2$ when $G = \text{SO}(2r+1)$ and α is a short root and $\varepsilon_{\alpha,G}^\vee = 1$ in all other cases.

(b) $\varepsilon_{\alpha,G}$ is characterized by the property that $\alpha/\varepsilon_{\alpha,G}$ is a primitive vector in Λ^\vee .

Proof. (a) This is standard and in fact the statement for $\varepsilon_{\alpha,G}^\vee$ was already noted in [DG02]. The explicit argument goes as follows. Without loss of generality we may assume that α is a simple root. For any group G we have $\Lambda_G \supset \text{corts}_{\mathfrak{g}}$, so for any root β we get $(\alpha, \beta^\vee) \in (\alpha, \Lambda_G)$. When β is simple we can read this number from the Dynkin diagram:

$$(\alpha, \beta^\vee) = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = \begin{cases} 2 & \text{when } \alpha = \beta, \\ -n & \text{when } \beta \text{ is short and } n \text{ edges connect } \alpha \text{ and } \beta, \\ -1 & \text{when } \beta \text{ is long and connected to } \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $\varepsilon_{\alpha,G} = 1$ unless all roots β connected to α are short and connect to α by an even number of edges. This happens only when α is a long root and \mathfrak{g} is of type \mathbf{C}_r . In the latter case we compute that $(\alpha, \text{corts}_{\mathfrak{g}}) = 2\mathbb{Z}$, while $(\alpha, \text{cowts}_{\mathfrak{g}}) = \mathbb{Z}$.

(b) Clearly if k is an integer and $\alpha/k \in \Lambda^\vee$, then k divides $\varepsilon_{\alpha,G}$. So we only need to check that for a long root α of $Sp(r)$ we have that $\alpha/2$ is in the weight lattice. But the root lattice for type C_r has generators $e_1 - e_2, \dots, e_{r-1} - e_r, 2e_r$ and the weight lattice has generators e_1, e_2, \dots, e_r . Again, up to a W -action, we may assume that α is a simple root, i.e. that $\alpha = 2e_r$. Thus $\alpha/2 = e_r$ which is indeed in the weight lattice. \square

Consider the cover $p : \tilde{C} \rightarrow C$. We will denote the branch locus of this cover by $S = \{s_1, \dots, s_b\} \subset C$. For each $i = 1, \dots, b$ we will write α_i for the root of \mathfrak{g} determined (up to W action) by s_i . Let $\varepsilon_i := \varepsilon_{\alpha_i,G}$ and $\varepsilon_i^\vee := \varepsilon_{\alpha_i,G}^\vee$. We write $j : U \hookrightarrow C$ for the inclusion of the complement, and $p^\circ : p^{-1}(U) \rightarrow U$ for the unramified part of p . Define a local system A on U by $A := (p_*^o \Lambda)^W$. Note that the fibers of A are non-canonically isomorphic to Λ . We can also consider ${}^L A = (p_*^o ({}^L \Lambda))^W$. The canonical identification ${}^L \Lambda = \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z})$ gives also an identification ${}^L A = A^\vee = \underline{\text{Hom}}(\Lambda, \mathbb{Z})$.

Lemma 2.3 *There are natural isomorphisms of sheaves $\mathcal{T}_{\mathbb{R}}^o \cong (j_* A) \otimes S^1$ and $\overline{\mathcal{T}}_{\mathbb{R}} \cong j_*(A \otimes S^1)$, while $\mathcal{T}_{\mathbb{R}}$ is determined by the commutative diagram:*

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \rightarrow & \mathcal{T}_{\mathbb{R}}^o & \longrightarrow & \mathcal{T}_{\mathbb{R}} & \longrightarrow & \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i & \rightarrow 0 \\
& \parallel & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathcal{T}_{\mathbb{R}}^o & \longrightarrow & \overline{\mathcal{T}}_{\mathbb{R}} & \xrightarrow{\xi} & \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee & \rightarrow 0 \\
& & & \downarrow & & \downarrow \epsilon^\vee & \\
& & & \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i^\vee & = & \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i^\vee & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

Proof. Clearly the sheaves $\mathcal{T}_{\mathbb{R}}^o$, $\overline{\mathcal{T}}_{\mathbb{R}}$, $(j_* A) \otimes S^1$ and $j_*(A \otimes S^1)$ coincide on U . Since A is a local system we have (see Section 4.1) $(j_* A)_{s_i} \cong \Lambda^{\rho_i}$, where $\rho_i(\lambda) = \lambda - (\alpha_i, \lambda) \alpha_i^\vee$ is the reflection corresponding to α_i . Similarly $(j_*(A \otimes S^1))_{s_i} \cong (\Lambda \otimes S^1)^{\rho_i}$. The formula for the reflection ρ_i and (1) now imply that $\mathcal{T}_{\mathbb{R}, s_i}^o = (j_* A)_{s_i} \otimes S^1$ and $\overline{\mathcal{T}}_{\mathbb{R}, s_i} = (j_*(A \otimes S^1))_{s_i}$.

On the stalk at s_i the map ξ is given by

$$\xi(\lambda \otimes z) := z^{(\alpha_i/\varepsilon_i, \lambda)} \in \mu_{\varepsilon_i \varepsilon_i^\vee} \subset S^1.$$

Here $\mu_{\varepsilon_i \varepsilon_i^\vee} \subset S^1$ denotes the roots of unity of order $\varepsilon_i \varepsilon_i^\vee$. Since $\varepsilon_i \varepsilon_i^\vee$ divides 2, we have a natural identification $\mu_{\varepsilon_i \varepsilon_i^\vee} = \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee$.

From (1) we now deduce that $\mathcal{T}_{\mathbb{R}}^o = \ker(\xi)$, and that $\mathcal{T}_{\mathbb{R}} = \ker(\epsilon^\vee \circ \xi)$, where

$$\epsilon^\vee : \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^\vee \rightarrow \bigoplus_{i=1}^b \mathbb{Z}/\varepsilon_i$$

is the map which multiplies the i -th summand by ε_i^\vee . \square

Claim 2.4 (i) *The connected components P^o , P , \overline{P} of $H^1(C, \mathcal{T}^o)$, $H^1(C, \mathcal{T})$, $H^1(C, \overline{\mathcal{T}})$ are abelian varieties. The natural maps $H^1(C, \mathcal{T}^o) \rightarrow H^1(C, \mathcal{T}) \rightarrow H^1(C, \overline{\mathcal{T}})$ and $P^o \rightarrow P \rightarrow \overline{P}$ are surjective.*

(ii) *The group of connected components of $H^1(C, \mathcal{T}^o)$ is $\mathbb{Z}/2$ for $G = \mathrm{Sp}(r)$ and is $\pi_1(G)$ otherwise.*

(iii) *The group of connected components of $H^1(C, \mathcal{T})$ is always $\pi_1(G)$, so the components of the fiber of $h : \mathbf{Higgs} \rightarrow B$ are in one-to-one correspondence with the components of the G -Hitchin \mathbf{Higgs} system itself.*

Proof. (i) We already noted that the connected component of $H^1(C, \overline{\mathcal{T}})$ is an abelian variety, and that $\overline{\mathcal{T}}/\mathcal{T}$ and $\mathcal{T}/\mathcal{T}^o$ have finite supports and fibers which are finite groups. It follows that $H^1(C, \mathcal{T}^o)$ and $H^1(C, \mathcal{T})$ map to $H^1(C, \overline{\mathcal{T}})$ surjectively with finite kernels. In particular the connected components of $H^1(C, \mathcal{T}^o)$ and $H^1(C, \mathcal{T})$ are also abelian varieties.

(ii) Consider the exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1 \rightarrow 0$ of constant sheaves on C . Tensoring with j_*A gives

$$\underline{\mathrm{Tor}}_1(j_*A, S^1) \xrightarrow{\partial} j_*A \longrightarrow (j_*A) \otimes \mathbb{R} \longrightarrow \mathcal{T}^o \longrightarrow 0.$$

The sheaf $\underline{\mathrm{Tor}}_1(j_*A, S^1)$ is supported on S while j_*A has no compactly supported sections. Therefore $\partial = 0$ and we get a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(j_*A \otimes S^1)/H^1(j_*A) & \longrightarrow & H^1(C, \mathcal{T}_{\mathbb{R}}^o) & \longrightarrow & H^2(C, j_*A) \longrightarrow 0. \\ & & \parallel & & & & \\ & & H^1(C, \mathcal{T}_{\mathbb{R}}^o)^o & & & & \end{array}$$

The group of connected components of $H^1(C, \mathcal{T}_{\mathbb{R}}^o)$ is therefore $H^2(C, j_*A)$, which can be identified (see Lemma 4.4) with $H^1(U, A^\vee)_{\mathrm{tor}}^\wedge$.

As we will see in Corollary 4.7

$$H^1(U, A^\vee)_{\mathrm{tor}} = \left(\frac{(\Lambda^\vee)^b}{(1 - \rho_1, 1 - \rho_2, \dots, 1 - \rho_b)\Lambda^\vee} \right)_{\mathrm{tor}}.$$

For any inclusion of lattices $N \subset M$, the torsion in M/N is equal to the quotient N'/N where $N' := \{m \in M \mid k \cdot m \in N \text{ for some } k \neq 0 \in \mathbb{Z}\}$ is the saturation of N in M . In our case $N = \Lambda^\vee$, while the saturation is

$$N' = \{\xi \in \mathfrak{t}^\vee \mid (\xi, \alpha^\vee) \cdot \alpha \in \Lambda^\vee \text{ for every root } \alpha\}.$$

This holds since our genericity assumption on \tilde{C} implies that $D^\alpha \subset \tilde{C}$ is non-empty for every root α . Using the characterization of $\varepsilon_{\alpha, G}$ in Lemma 2.2(b) we see that $\xi \in N'$ if and only if $\varepsilon_{\alpha, G}(\xi, \alpha^\vee) \in \mathbb{Z}$ for all roots α . In case $G = \mathrm{Sp}(r)$ we see that N' contains the weight lattice

as a sublattice of index two. Explicitly the weight lattice is generated by e_1, \dots, e_r and N' is spanned by the e_i 's and the additional vector $\frac{1}{2} \sum_{i=1}^r e_i$. For all other G , all ε 's are 1, so N' is the weight lattice. We conclude that

$$H^1(U, A^\vee)_{\text{tor}} = \begin{cases} \mathbb{Z}/2 & \text{when } G = \text{Sp}(r) \\ \text{wts}_{\mathfrak{g}} / \Lambda_G^\vee = \pi_1(G)^\wedge & \text{for all other } G. \end{cases}$$

This completes the proof of (ii).

(iii) As we saw in Lemma 2.3 we have $\mathcal{T}^0 = \mathcal{T}$ except when $G = \text{Sp}(r)$. In the latter case the fiber of the Hitchin map was shown to be connected in [Hit87], using an interpretation via spectral covers. \square

In general for any compact torus H we define the cocharacter lattice $\mathbf{cochar}(H)$ as the lattice of homomorphisms from the circle S^1 to H . We recover H as $\mathbf{cochar}(H) \otimes S^1$.

Claim 2.5 (i) *There is a natural isomorphism $\mathbf{cochar}(P^o) = H^1(C, j_* A)_{\text{tf}}$.*

(i) *There is a natural isomorphism $\mathbf{cochar}(\overline{P}) = H^1(C, j_* A^\vee)^\vee$.*

(iii) *The map $\zeta : \mathbf{cochar}(\overline{P}) \rightarrow \bigoplus_{i=1}^b \mathbb{Z} / \varepsilon_i \varepsilon_i^\vee$ induced from the map ξ in Lemma 2.3 satisfies*

$$\begin{aligned} \ker(\zeta) &= \mathbf{cochar}(P^o) \\ \ker(\epsilon^\vee \circ \zeta) &= \mathbf{cochar}(P). \end{aligned}$$

Proof. (i) By Lemma 2.3 we know that $\mathcal{T}_{\mathbb{R}}^o = (j_* A) \otimes S^1$. As in the proof of Claim 2.4 (ii), we tensor the exponential sequence for S^1 by $j_* A$ and we get

$$\underline{\text{Tor}}_1(j_* A, S^1) \xrightarrow{\partial} j_* A \longrightarrow j_* A \otimes \mathbb{R} \longrightarrow \mathcal{T}_{\mathbb{R}}^o \longrightarrow 0.$$

Again the sheaf $\underline{\text{Tor}}_1(j_* A, S^1)$ is supported on S while $j_* A$ has no compactly supported sections. Therefore $\partial = 0$ and we get a short exact sequence

$$0 \longrightarrow H^1(j_* A) \otimes S^1 \longrightarrow H^1(C, \mathcal{T}^o) \longrightarrow H^2(C, j_* A) \longrightarrow 0.$$

Since $H^2(C, j_* A)$ is finite and $H^1(j_* A) \otimes S^1$ is connected, it follows that $P^o = H^1(j_* A) \otimes S^1$, or equivalently $\mathbf{cochar}(P^o) = H^1(C, j_* A)_{\text{tf}}$.

(ii) Start with the Leray spectral sequence (aka Mayer-Vietoris) for the inclusion $j : U \subset C$ and the sheaf A . It gives

$$0 \rightarrow H^1(C, j_* A) \rightarrow H^1(U, A) \rightarrow Q \rightarrow 0,$$

where $Q = \ker(H^0(R^1 j_* A) \rightarrow H^2(j_* A))$ (see Section 4.1 for details). We tensor this sequence with S^1 and map to the Leray sequence for j and $A \otimes S^1$:

$$\begin{array}{ccccccc} Q_{\text{tor}} & \longrightarrow & H^1(C, j_* A) \otimes S^1 & \longrightarrow & H^1(U, A) \otimes S^1 & \longrightarrow & Q \otimes S^1 \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & H^1(C, \overline{\mathcal{T}}_{\mathbb{R}}) & \longrightarrow & H^1(U, A \otimes S^1) & \longrightarrow & Q \otimes S^1 \longrightarrow 0. \end{array}$$

Recall that \overline{P} is the connected component of $H^1(C, \overline{\mathcal{T}}_{\mathbb{R}})$ and that by Lemma 2.3 we have $\overline{\mathcal{T}}_{\mathbb{R}} = j_*(A \otimes S^1)$. It follows that \overline{P} can be identified with the image $\text{im}[P^o \rightarrow H^1(U, A) \otimes S^1]$. In particular, on character lattices we get

$$\text{char } \overline{P} = \text{im}[H^1(U, A)^{\vee} \rightarrow \text{char } P^o = H^1(C, j_* A)^{\vee}] = H^1(C, j_* A^{\vee})_{\text{tf}},$$

where the last equality follows from Corollary 4.5.

(iii) This is immediate from parts (i) and (ii) and the commutative diagram of sheaves in Lemma 2.3. \square

The statement of the previous claim can be organized in a diagram:

$$\begin{array}{ccccc} H^1(C, j_* A)_{\text{tf}} & \subset & \text{cochar}(P) & \subset & H^1(C, j_* A^{\vee})^{\vee} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \subset & \oplus_{i=1}^b \mathbb{Z}/\varepsilon_i^{\vee} & \subset & \oplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^{\vee}. \end{array}$$

Writing the analogous diagram for ${}^L G$ and dualizing gives

$$\begin{array}{ccccc} H^1(C, j_* A)_{\text{tf}} & \subset & \text{cochar}({}^L P)^{\vee} & \subset & H^1(C, j_* A^{\vee})^{\vee} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \subset & \oplus_{i=1}^b \mathbb{Z}/\varepsilon_i^{\vee} & \subset & \oplus_{i=1}^b \mathbb{Z}/\varepsilon_i \varepsilon_i^{\vee}. \end{array}$$

This gives the desired isomorphism $\text{cochar}(P) \cong \text{cochar}({}^L P)^{\vee}$. As noted in Corollary 4.5 this isomorphism is compatible with the Poincaré duality map for the cohomologies of A and A^{\vee} on U . Finally, the Leray spectral sequence for $p : \tilde{C} \rightarrow C$ allows us to identify the universal covers of P and ${}^L P$ with the complex vector spaces $H^1(\tilde{C}, \Lambda \otimes \mathcal{O}_{\tilde{C}})^W$ and $H^1(\tilde{C}, \Lambda^{\vee} \otimes \mathcal{O}_{\tilde{C}})^W$. In particular, the isomorphism $\text{cochar}(P) \cong \text{cochar}({}^L P)^{\vee}$ is compatible with the Serre duality isomorphism $H^1(\tilde{C}, \Lambda \otimes \mathcal{O}_{\tilde{C}})$ and $H^1(\tilde{C}, \Lambda^{\vee} \otimes \mathcal{O}_{\tilde{C}})^{\vee}$ and hence induces an isomorphism between the polarized abelian varieties \hat{P} and ${}^L P$. \square

3 Duality for Higgs gerbes

In this section we extend the duality of cameral Pryms established in Theorem A to a more general duality for the stacks of Higgs bundles, considered as families of stacky groups over the Hitchin base.

3.1 Triviality

The moduli stack of G -Higgs bundles on C was defined and studied in detail in [DG02]. We briefly recall the highlights of that discussion. Let

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{p} & B \times C \\ & \searrow \tilde{\pi} & \swarrow \pi \\ & B & \end{array}$$

denote the universal cameral cover. In [DG02] the authors introduced a sheaf \mathcal{T} of abelian groups on $B \times C$, defined as

$$\mathcal{T}(U) = \left\{ t \in \Gamma \left(p^{-1}(U), \Lambda \otimes \mathcal{O}_{\tilde{\mathcal{C}}}^{\times} \right)^W \mid \begin{array}{l} \text{for every root } \alpha \text{ of } \mathfrak{g} \\ \text{we have } \alpha(t)|_{D^{\alpha}} = 1 \end{array} \right\}.$$

It was shown in [DG02] that relatively over the Hitchin base the stack **Higgs** of Higgs bundles is a banded \mathcal{T} -gerbe. Informally, this means that the sheaf of groups for which **Higgs** is a gerbe is \mathcal{T} itself rather than a more general sheaf of groups which is only locally isomorphic to \mathcal{T} . Equivalently, when viewed as a stack over the Hitchin base, **Higgs** is a torsor over the commutative group stack **Tors** $_{\mathcal{T}}$ parametrizing \mathcal{T} -torsors along the fibers of $\pi : B \times C \rightarrow B$. This description is valid for Higgs bundles over a base variety of arbitrary dimension. When the base is a compact curve, the picture can be made even more precise.

Lemma 3.1 *Let C be a smooth compact curve and let **Higgs** be the moduli stack of G -Higgs bundles on C .*

- (a) *The commutative group stack **Tors** $_{\mathcal{T}}$ parametrizing the \mathcal{T} -torsors along the fibers of π is isomorphic to the Picard stack associated (see [SGA, Section 1.4 of Exposé XVIII]) with the amplitude one complex $R^{\bullet}\pi_{*}\mathcal{T}[1]$ of abelian sheaves on B .*
- (b) *There exists an isomorphism $\mathbf{Higgs} \cong \mathbf{Tors}_{\mathcal{T}}$ of stacks over B .*

Proof. (a) This follows from the fact $\pi : B \times C \rightarrow B$ is smooth of relative dimension one, the standard description of torsors in terms of Čech cocycles, and the definition (see [SGA, Section 1.4 of Exposé XVIII]) of a Picard stack associated with an amplitude one complex of abelian sheaves.

(b) By [DG02, Theorem 4.4] the stack **Higgs** is a torsor over the commutative group stack **Tors** $_{\mathcal{T}}$. Thus to get the isomorphism $\mathbf{Higgs} \cong \mathbf{Tors}_{\mathcal{T}}$, it suffices to show that the stacky Hitchin fibration $\mathbf{h} : \mathbf{Higgs} \rightarrow B$ admits a section. This is due to Hitchin who in [Hit92] constructed a family of holomorphic sections of $\mathbf{h} : \mathbf{Higgs} \rightarrow B$ induced from a Kostant section of the Chevalley map $\mathfrak{g} \rightarrow \mathfrak{g}/G \cong \mathfrak{t}/W$. Since these sections play a prominent role in what follows, we briefly recall Hitchin's construction.

Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \subset \mathfrak{g}$ be any principal \mathfrak{sl}_2 triple in \mathfrak{g} . This means that $\mathbf{e}, \mathbf{f}, \mathbf{g}$ span a Lie subalgebra in \mathfrak{g} isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, and that \mathbf{e} and \mathbf{f} are regular nilpotent elements of

\mathfrak{g} . Let $\mathfrak{C}(e) \subset \mathfrak{g}$ be the centralizer of the element e in the algebra \mathfrak{g} . Consider the linear coset $\mathbf{f} + \mathfrak{C}(e) = \{\mathbf{f} + x | x \in \mathfrak{C}(e)\}$. In [Kos63] Kostant showed that the map $\mathfrak{g} \rightarrow \mathfrak{t}/W$ becomes an isomorphism, when restricted to $\mathbf{f} + \mathfrak{C}(e)$. Thus $\mathbf{f} + \mathfrak{C}(e)$ is a section for the Chevalley projection. By construction this section consists of regular elements in \mathfrak{g} and is a generalization of the rational canonical form of a matrix.

Fix a Kostant section $\mathbf{k} : \mathfrak{t}/W \rightarrow \mathfrak{g}$, corresponding to an \mathfrak{sl}_2 -triple in \mathfrak{g} . The inclusion of the \mathfrak{sl}_2 -triple in \mathfrak{g} induces a group homomorphism $\varrho : \mathrm{SL}_2(\mathbb{C}) \rightarrow G$. Let $\zeta \in \mathrm{Pic}^{g-1}(C)$, $\zeta^{\otimes 2} = K_C$ be a theta characteristic on C . Consider the frame bundle $\underline{\mathrm{Isom}}(\zeta \oplus \zeta^{-1}, \mathcal{O}^{\oplus 2})$ of the vector bundle $\zeta \oplus \zeta^{-1}$ on C . This is a principal $\mathrm{SL}_2(\mathbb{C})$ -bundle which via ϱ gives rise to an associated principal G bundle $P := \underline{\mathrm{Isom}}(\zeta \oplus \zeta^{-1}, \mathcal{O}^{\oplus 2}) \times_{\varrho} G$ on C . Recall that the Hitchin base B is the space of sections of the bundle $(K_C \otimes \mathfrak{t})/W$ on C . Let \mathbf{U} denote the total space of the bundle $(K_C \otimes \mathfrak{t})/W$, and let $u : \mathbf{U} \rightarrow C$ be the natural projection. We have

$$\mathrm{ad}(u^*P) = u^* \mathrm{ad}(P) = u^* \underline{\mathrm{Isom}}(\zeta \oplus \zeta^{-1}, \mathcal{O}^{\oplus 2}) \times_{\mathrm{ad}(\varrho)} \mathfrak{g}.$$

In [Hit92] Hitchin checked that the Kostant section $\mathbf{k} : \mathfrak{t}/W \rightarrow \mathfrak{g}$ induces a well defined section $\varphi \in H^0(\mathbf{U}, \mathrm{ad}(u^*P) \otimes u^*K_C)$ and hence a u^*K_C -valued Higgs bundle (u^*P, φ) on \mathbf{U} . Pulling back this Higgs bundle by the sections $b \in B = H^0(C, \mathbf{U})$, one gets a family of Higgs bundles on C , parameterized by B . We will call the resulting section of $\mathbf{h} : \mathbf{Higgs} \rightarrow B$ the *Hitchin section* and denote it by $\mathbf{v} : B \rightarrow \mathbf{Higgs}$. \square

3.2 Stabilizers, components, and universal bundles

From now on, we restrict our attention to the open substack of \mathbf{Higgs} consisting of stable G -Higgs bundles whose automorphism group is the minimal possible, i.e. coincides with the center of G . The Hitchin fiber for a cameral cover in $B - \Delta$ consists only of stable Higgs bundles, and in fact each Higgs bundle in such fiber has minimal automorphism group:

Lemma 3.2 (i) $\mathbf{Higgs}_{|B-\Delta}$ is a smooth Deligne-Mumford stack with a coarse moduli space $\mathbf{Higgs}_{|B-\Delta}$. If we view $\mathbf{Higgs}_{|B-\Delta}$ as a group stack, then the group of connected components, as well as the connected components of each fiber of $\mathbf{h} : \mathbf{Higgs}_{|B-\Delta} \rightarrow (B - \Delta)$ are canonically isomorphic to $\pi_1(G)$.

(ii) $\mathbf{Higgs}_{|B-\Delta}$ is a banded $Z(G)$ -gerbe over $\mathbf{Higgs}_{|B-\Delta}$ which is locally trivial over $B - \Delta$. In particular the restriction of $\mathbf{Higgs}_{|B-\Delta}$ to a Hitchin fiber is a trivial gerbe.

(iii) The gerbe $\mathbf{Higgs}_{|B-\Delta} \rightarrow \mathbf{Higgs}_{|B-\Delta}$ measures the obstruction to lifting the universal G_{ad} -Higgs bundle to a universal G -Higgs bundle.

Proof. (i) It is well known [Sim94, Sim95] that the stack \mathbf{Higgs} of G -Higgs bundles is an Artin algebraic stack with an affine diagonal which is locally of finite type. The substack of semistable Higgs bundles is of finite type and has a quasi-projective moduli space. The statement about the connected components is now automatic, since $\pi_0(\mathbf{Higgs}_{|B-\Delta}) =$

$\pi_0(\mathbf{Higgs}_{|B-\Delta})$ and the connected components of $\mathbf{Higgs}_{|B-\Delta}$ and the corresponding Hitchin fibers were already described in Claim 2.4(ii).

It is also known [Sim95, BD03] that the stack \mathcal{Higgs} is a local complete intersection which is smooth at all points with finite stabilizers. Therefore it suffices to show that $\mathcal{Higgs}_{|B-\Delta}$ parametrizes Higgs bundles with minimal automorphism group. In [Fal93, Theorem III.2] Faltings showed that B contains a Zariski open and dense subset $B^o \subset B$, such that $\mathcal{Higgs}_{|B^o}$ parametrizes only stable Higgs bundles with automorphism group $Z(G)$. We will give a direct argument for this over $B - \Delta$, i.e. we will show that $B^o \supset B - \Delta$.

Fix a point $b \in (B - \Delta)$ and let $p : \tilde{C} \rightarrow C$ be the corresponding cameral cover. We must show that every object in the groupoid $\mathcal{Higgs}_{\tilde{C}} := \mathbf{h}^{-1}(b)$ has automorphism group $Z(G)$. As explained in Lemma 3.1, $\mathcal{Higgs}_{\tilde{C}}$ is the groupoid of $\mathcal{T}_{|\tilde{C}}$ -torsors, and hence the automorphism group of any object in $\mathcal{Higgs}_{\tilde{C}}$ is isomorphic to the cohomology group $H^0(\tilde{C}, \mathcal{T}_{\tilde{C}})$. By the argument we used in the proof of Theorem A(2) we have isomorphisms $H^0(\tilde{C}, \mathcal{T}) \cong H^0(\tilde{C}, \mathcal{T}_{\mathbb{R}})$, $H^0(\tilde{C}, \mathcal{T}^o) \cong H^0(\tilde{C}, \mathcal{T}_{\mathbb{R}}^o)$, $H^0(\tilde{C}, \overline{\mathcal{T}}) \cong H^0(\tilde{C}, \overline{\mathcal{T}}_{\mathbb{R}})$. Thus it suffices to compute $H^0(\tilde{C}, \mathcal{T}_{\mathbb{R}})$. We start by calculating the global sections of $\mathcal{T}_{\mathbb{R}}^o = (j_*A) \otimes S^1$. As in the proof of Claim 2.4(ii) we get a short exact sequence of sheaves on \tilde{C} :

$$0 \rightarrow j_*A \rightarrow (j_*A) \otimes \mathbb{R} \rightarrow (j_*A) \otimes S^1 \rightarrow 0.$$

Passing to cohomology, and taking into account that $H^0(\tilde{C}, j_*A) = 0$ and $H^0(\tilde{C}, (j_*A) \otimes \mathbb{R}) = H^0(\tilde{C}, j_*A) \otimes \mathbb{R} = 0$, we get that

$$\begin{aligned} H^0(\tilde{C}, \mathcal{T}_{\mathbb{R}}^o) &= \ker \left[H^1(\tilde{C}, j_*A) \rightarrow H^1(\tilde{C}, j_*A \otimes \mathbb{R}) \right] \\ &= H^1(\tilde{C}, j_*A)_{\text{tor}}. \end{aligned}$$

The latter group can be calculated explicitly from Corollary 4.7. In the notation of Corollary 4.7, let N denote the saturation of $(1 - \rho_1, \dots, 1 - \rho_b)\Lambda$ inside $\oplus_{i=1}^b \mathbb{Z}\varepsilon_i\alpha_i^\vee$. Then

$$\begin{aligned} H^1(\tilde{C}, j_*A)_{\text{tor}} &= N/(1 - \rho_1, \dots, 1 - \rho_b)\Lambda \\ &= \{ \xi \in \mathfrak{t} \mid (\xi, \alpha) \in \varepsilon_{\alpha, G}\mathbb{Z} \text{ for every root } \alpha \} / \Lambda \\ &= \begin{cases} Z(G) & \text{if } G \neq \text{Sp}(r) \\ 0 & \text{if } G = \text{Sp}(r). \end{cases} \end{aligned}$$

From Lemma 2.3 we know that as long as $G \neq \text{Sp}(r)$ we have $\mathcal{T}_{\mathbb{R}}^o = \mathcal{T}_{\mathbb{R}}$. This proves our claim for $G \neq \text{Sp}(r)$. For $G = \text{Sp}(r)$, Lemma 2.3 gives a short exact sequence

$$0 \rightarrow \mathcal{T}_{\mathbb{R}}^o \rightarrow \mathcal{T}_{\mathbb{R}} \rightarrow \oplus_{i=1}^b \mathbb{Z}/\varepsilon_i \rightarrow 0,$$

and after passing to cohomology we get

$$\begin{aligned} H^0(\tilde{C}, \mathcal{T}_{\mathbb{R}}) &= \ker \left[\oplus_{i=1}^b \mathbb{Z}/\varepsilon_i \rightarrow H^1(\tilde{C}, \mathcal{T}_{\mathbb{R}}^o) \right] \\ &= Z(\text{Sp}(r)) \cong \mathbb{Z}/2, \end{aligned}$$

where $Z(\mathrm{Sp}(r)) \cong \mathbb{Z}/2$ maps diagonally in $\oplus_{i=1}^b \mathbb{Z}/\varepsilon_i$. This proves our assertion about the automorphisms of objects in $\mathcal{Higgs}_{\tilde{C}}$ and finishes the proof of (i).

(ii) As we saw above, the \mathcal{T} torsors in $\mathcal{Higgs}_{|B-\Delta}$ all have automorphism groups isomorphic to $Z(G)$, and so $\mathcal{Higgs}_{|B-\Delta}$ is a $Z(G)$ -gerbe on $\mathbf{Higgs}_{|B-\Delta}$. In particular $R^0\pi_*\mathcal{T}$ is a local system on $B - \Delta$ with fiber $Z(G)$. However $Z(G) = T^W \subset T$ and so, by the definition of \mathcal{T} we have a canonical inclusion of the constant sheaf $Z(G)$ into \mathcal{T} . Thus, every element in $Z(G)$ gives rise to a global section of \mathcal{T} on $B \times C$, and hence to a global section of $R^0\pi_*\mathcal{T}$ on $B - \Delta$. This shows that $R^0\pi_*\mathcal{T}$ is the constant sheaf and so $\mathcal{Higgs}_{|B-\Delta}$ is banded as a gerbe over $\mathbf{Higgs}_{|B-\Delta}$. Finally, note that locally over $B - \Delta$, the universal cameral cover $\tilde{\mathcal{C}}$ admits a section. The stack of \mathcal{T} -torsors which are framed along such a section is isomorphic to the space \mathbf{Higgs} , which shows that the gerbe $\mathcal{Higgs}_{|B-\Delta}$ is locally trivial over $(B - \Delta)$.

(iii) This is completely analogous to the $\mathrm{PSL}(r)$ argument in [HT03]. Let G_{ad} be the adjoint form of G . From part (ii) it follows that the stack of G_{ad} -Higgs bundles that have cameral cover in $B - \Delta$ is actually a space, i.e. $\mathcal{Higgs}_{G_{\mathrm{ad}}|(B-\Delta)} = \mathbf{Higgs}_{G_{\mathrm{ad}}|(B-\Delta)}$. Since the stack always has a universal bundle, we have a universal G_{ad} -Higgs bundle (V, φ) on $\mathbf{Higgs}_{G_{\mathrm{ad}}|(B-\Delta)} \times C$. The natural map $G \rightarrow G_{\mathrm{ad}}$ induces a morphism of spaces $q : \mathbf{Higgs}_{|B-\Delta} \rightarrow \mathbf{Higgs}_{G_{\mathrm{ad}}|(B-\Delta)}$ and we can consider the pullback G_{ad} -Higgs bundle $q^*(V, \varphi)$ on $\mathbf{Higgs}_{|B-\Delta}$. Since $\ker[G \rightarrow G_{\mathrm{ad}}] = Z(G)$, it follows that the obstruction to lifting $q^*(V, \varphi)$ to a G -Higgs bundle is simply the obstruction to the existence of an universal G -Higgs bundle on $\mathbf{Higgs}_{|(B-\Delta)} \times C$, i.e it is the gerbe $\mathcal{Higgs}_{|(B-\Delta)} \times C$. In particular, restricting to $\mathbf{Higgs}_{|(B-\Delta)} \times \{\mathrm{pt}\}$ we get the statement (iii). \square

3.3 Global duality

We are now ready to state the main result of this section. For any commutative group stack \mathcal{X} over $B - \Delta$ with zero section $\mathfrak{v} : (B - \Delta) \rightarrow \mathcal{X}$, we define the dual commutative group stack as

$$\mathcal{X}^D := \underline{\mathrm{Hom}}_{\mathrm{gp}}(\mathcal{X}, \mathcal{O}^\times[1]).$$

Geometrically \mathcal{X}^D is the stack of group extensions of \mathcal{X} by \mathcal{O}^\times , or equivalently, the stack parameterizing pairs $(\mathcal{L}, \mathfrak{f})$, where \mathcal{L} is a translation invariant line bundle on \mathcal{X} , and $\mathfrak{f} : \mathfrak{v}^*\mathcal{L} \rightarrow \mathcal{O}$ is a trivialization of \mathcal{L} along the zero section of \mathcal{X} .

With this notation we now have:

Theorem B *Let \mathcal{Higgs} be the stack of G Higgs bundles on a curve C and let ${}^L\mathcal{Higgs}$ be the stack of ${}^L G$ Higgs bundles on C . Use the isomorphism $\mathbf{l} : B \rightarrow {}^L B$ from **Theorem A(1)** to identify $B - \Delta$ with ${}^L B - {}^L \Delta$. Under this identification one has a canonical isomorphism*

$$(3) \quad ({}^L\mathcal{Higgs}_{|B-\Delta})^D \cong \mathcal{Higgs}_{|B-\Delta}$$

of commutative group stacks over $B - \Delta$.

Proof. Here we only consider the parts of the stacks of Higgs bundles sitting over $B - \Delta$. To simplify notation throughout the proof we will write \mathbf{Higgs} and ${}^L\mathbf{Higgs}$ instead of $\mathbf{Higgs}_{|B-\Delta}$ and ${}^L\mathbf{Higgs}_{|B-\Delta}$. We will use the abelianization of Higgs bundles described in [DG02, Section 6] to define Hecke operators on the moduli stack of Higgs bundles. More precisely, for every $\mu \in \Lambda$ and $x \in \tilde{C} \subset \tilde{\mathcal{C}}$ we will construct a canonical Hecke automorphism $\mathfrak{H}_{\mu,x} : \mathbf{Higgs}_{\tilde{C}} \rightarrow \mathbf{Higgs}_{\tilde{C}}$. These automorphisms can be combined into a single map of stacks

$$(4) \quad \mathfrak{H} : \mathbf{Higgs} \times_{(B-\Delta)} \tilde{\mathcal{C}} \times \Lambda \rightarrow \mathbf{Higgs},$$

which we will call *the abelianized Hecke correspondence*. This map induces a natural map on coarse moduli spaces which we will denote again by \mathfrak{H} .

Let (V, φ) be a K_C -valued G -Higgs bundle with cameral cover $p : \tilde{C} \rightarrow C$, and let $x \in \tilde{C}$, $\lambda \in \Lambda$. Informally \mathfrak{H} takes the data $((V, \varphi), x, \lambda)$ to a new Higgs bundle (V', φ') having the same cameral cover $p : \tilde{C} \rightarrow C$, an underlying G -bundle V' which is a modification of V at $p(x)$ in the direction λ , and a Higgs field φ' which agrees with the original φ on $C - \{p(x)\}$.

More formally, [DG02, Theorem 6.4] establishes an equivalence between the groupoid of G -Higgs bundles on C with cameral cover \tilde{C} and a collection of data on \tilde{C} consisting of a ramification twisted W -invariant T -bundle $\mathcal{L} = \mathcal{L}_{(V, \varphi)}$ on \tilde{C} and some additional framing structures satisfying compatibility conditions. Now the point $x \in \tilde{C}$ determines the degree one $\mathcal{O}_{\tilde{C}}^\times$ -bundle $\mathcal{O}_{\tilde{C}}^\times(x)$, and the cocharacter $\lambda : \mathbb{C}^\times \rightarrow T$ induces a T -bundle $\lambda(\mathcal{O}_{\tilde{C}}^\times(x))$ on \tilde{C} . This T -bundle is not W -invariant. Instead we need

$$S_{x,\lambda} := \bigotimes_{w \in W} (w\lambda)(\mathcal{O}_{\tilde{C}}(wx)),$$

which is a W -equivariant T -bundle. By definition the map \mathfrak{H} replaces the T -bundle \mathcal{L} by

$$\mathcal{L}' := \mathcal{L} \otimes_T S_{x,\lambda}$$

which is ramification twisted W -invariant, just like \mathcal{L} . The framing structures carry over trivially (i.e. by tensoring with the identity on $S_{x,\lambda}$) to \mathcal{L}' , and the necessary compatibilities are automatic as long as $x \in \tilde{C}$ is not a ramification point of $p : \tilde{C} \rightarrow C$. This defines the map \mathfrak{H} on the complement of the ramification in \tilde{C} and hence everywhere since we are working with a smooth curve.

Let now $\mathfrak{v} : (B - \Delta) \rightarrow \mathbf{Higgs}$ be a Hitchin section. By applying the Hecke map \mathfrak{H} to \mathfrak{v} we get Higgs bundle versions of the Abel-Jacobi map:

$$\begin{array}{ccc} & & \mathbf{Higgs} \\ & \nearrow \text{aj}^G & \downarrow \\ \tilde{\mathcal{C}} \times \Lambda & & \mathbf{Higgs} \\ & \searrow \text{aj}^G & \end{array}$$

We will use the Langlands dual version of this map to identify $({}^L\mathcal{Higgs})^D \cong \mathcal{Higgs}$. Fix a Hitchin section ${}^L\mathbf{v} : (B - \Delta) \rightarrow {}^L\mathcal{Higgs}$ and let

$$\mathbf{aj}^{LG} : \tilde{\mathcal{C}} \times \Lambda^\vee \rightarrow {}^L\mathcal{Higgs}$$

be the corresponding Abel-Jacobi map. If we view $({}^L\mathcal{Higgs})^D$ as the stack of translation invariant line bundles on ${}^L\mathcal{Higgs}$ which are framed along the Hitchin section ${}^L\mathbf{v}$, then pulling back via \mathbf{aj}^{LG} gives a map from $({}^L\mathcal{Higgs})^D$ to the stack of line bundles on $\tilde{\mathcal{C}} \times \Lambda^\vee$, or equivalently to the stack \mathcal{Bun}_T of T -bundles on $\tilde{\mathcal{C}}$.

If a translation invariant line bundle \mathcal{L} on ${}^L\mathcal{Higgs}$ comes from a translation invariant line bundle L on ${}^L\mathbf{Higgs}$ (i.e. if \mathcal{L} has weight zero on ${}^L\mathcal{Higgs}$), we know from Theorem A and translation invariance that the T -bundle $(\mathbf{aj}^{LG})^*\mathcal{L} = (\mathbf{aj}^{LG})^*L$ comes from a Higgs bundle in the neutral component \mathcal{Higgs}_0 . In general the weights of the line bundles on the $Z({}^LG)$ -gerbe ${}^L\mathcal{Higgs}$ are indexed by the group $Z({}^LG)^\wedge = \pi_1(G)$, i.e. by the connected components of \mathcal{Higgs} . If $x \in \tilde{\mathcal{C}}$ and a point $\lambda \in \Lambda$, note that the Hecke operator $\mathfrak{H}_{x,\lambda} : \mathcal{Higgs}_{\tilde{\mathcal{C}}} \rightarrow \mathcal{Higgs}_{\tilde{\mathcal{C}}}$ shifts the components of $\mathcal{Higgs}_{\tilde{\mathcal{C}}}$ by the image of λ in $\pi_1(G) = \Lambda / \text{coroot}_{\mathfrak{g}}$. In particular, any component of $\mathcal{Higgs}_{\tilde{\mathcal{C}}}$ can be reached from the neutral one by applying a suitable Hecke operator. So, to prove our theorem, it suffices to construct canonical translation invariant line bundles $\mathcal{L}_{x,\lambda}$ on ${}^L\mathcal{Higgs}_{\tilde{\mathcal{C}}}$ which make the following diagram strictly commutative:

$$\begin{array}{ccccc} ({}^L\mathcal{Higgs}_{\tilde{\mathcal{C}}})^D & \xrightarrow{(\mathbf{aj}^{LG})^*} & \mathcal{Bun}_T & \xleftarrow{\mathcal{L}_{(V,\varphi)} \leftarrow (V,\varphi)} & \mathcal{Higgs}_{\tilde{\mathcal{C}}} \\ \downarrow \otimes \mathcal{L}_{x,\lambda} & & \downarrow \otimes S_{x,\lambda} & & \downarrow \mathfrak{H}_{x,\lambda} \\ ({}^L\mathcal{Higgs}_{\tilde{\mathcal{C}}})^D & \xrightarrow{(\mathbf{aj}^{LG})^*} & \mathcal{Bun}_T & \xleftarrow{\mathcal{L}_{(V,\varphi)} \leftarrow (V,\varphi)} & \mathcal{Higgs}_{\tilde{\mathcal{C}}} \end{array}$$

In other words, we must find line bundles $\mathcal{L}_{x,\lambda}$ on ${}^L\mathcal{Higgs}$ such that the line bundle $(\mathbf{aj}^{LG})^*\mathcal{L}_{x,\lambda}$ on $\tilde{\mathcal{C}} \times \Lambda^\vee$, interpreted as a T -bundle on $\tilde{\mathcal{C}}$, should equal $S_{x,\lambda}$.

Next note that by Lemma 3.1, the choice of the Hitchin section ${}^L\mathbf{v}$ identifies ${}^L\mathcal{Higgs}$ with the group stack over $B - \Delta$ associated with $R^\bullet\pi_*({}^L\mathcal{T})[1]$, and similarly identifies ${}^L\mathbf{Higgs}$ with the commutative group scheme on $B - \Delta$ representing the sheaf $R^1\pi_*({}^L\mathcal{T})$. The sheaf

inclusion ${}^L\mathcal{T} \subset {}^L\overline{\mathcal{T}}$ induces natural maps of complexes of abelian sheaves on $B - \Delta$:

$$(5) \quad \begin{array}{ccc} R^\bullet \pi_*({}^L\mathcal{T}) & \longrightarrow & R^\bullet \pi_*({}^L\overline{\mathcal{T}}) \\ & & \parallel \\ & & R^\bullet \pi_* \left(\left[p_* \left(\Lambda^\vee \otimes \mathcal{O}_{\tilde{\mathcal{C}}}^\times \right) \right]^W \right) \\ & & \downarrow \\ & & R^\bullet \pi_* \left(p_* \left(\Lambda^\vee \otimes \mathcal{O}_{\tilde{\mathcal{C}}}^\times \right) \right) \\ & & \downarrow \text{Leray} \\ & & R^\bullet \tilde{\pi}_* \left(\Lambda^\vee \otimes \mathcal{O}_{\tilde{\mathcal{C}}}^\times \right) \\ & & \parallel \\ & & \left(R^\bullet \tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{C}}}^\times \right) \otimes \Lambda^\vee \\ & & \parallel \\ & & \mathbf{Pic}(\tilde{\mathcal{C}}/(B - \Delta)) \otimes \Lambda^\vee. \end{array}$$

$\searrow \scriptstyle L_{\mathbf{a}}$

The map labelled “Leray” comes from the Leray spectral sequence for $p : \tilde{\mathcal{C}} \rightarrow (B \times C)$ and is an isomorphism because p is finite.

The composite map $L_{\mathbf{a}} : R^\bullet \pi_*({}^L\mathcal{T}) \rightarrow \mathbf{Pic}(\tilde{\mathcal{C}}/B - \Delta) \otimes \Lambda^\vee$ induces a morphism of stacks

$$L_{\mathbf{a}} : {}^L\mathbf{Higgs} \rightarrow \mathbf{Pic}(\tilde{\mathcal{C}}/(B - \Delta)) \otimes \Lambda^\vee = \mathbf{Bun}_T,$$

where \mathbf{Bun}_T denotes the stack parametrizing T -bundles along the fibers of $\tilde{\pi} : \tilde{\mathcal{C}} \rightarrow (B - \Delta)$.

Similarly, if in diagram (5) we replace $R^\bullet \pi_*$ with $R^1 \pi_*$ we get a composite map $L_{\mathbf{a}} : R^1 \pi_*({}^L\mathcal{T}) \rightarrow \mathbf{Pic}(\tilde{\mathcal{C}}/B - \Delta) \otimes \Lambda^\vee$ which induces a morphism of spaces

$$L_{\mathbf{a}} : {}^L\mathbf{Higgs} \rightarrow \mathbf{Pic}(\tilde{\mathcal{C}}/(B - \Delta)) \otimes \Lambda^\vee = \mathbf{Bun}_T.$$

Combining these maps with the Abel-Jacobi maps for Higgs bundles we get commutative diagrams

$$(6) \quad \begin{array}{ccc} \tilde{\mathcal{C}} \times \Lambda^\vee & \xrightarrow{\text{aj}^{L_G}} & {}^L\mathbf{Higgs} \\ & & \downarrow L_{\mathbf{a}} \\ & & \mathbf{Pic}(\tilde{\mathcal{C}}/(B - \Delta)) \otimes \Lambda^\vee \\ & \searrow \scriptstyle \text{aj} \times \text{id} & \parallel \\ & & \mathbf{Bun}_T \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{\mathcal{C}} \times \Lambda^\vee & \xrightarrow{\text{aj}^{L_G}} & {}^L\mathbf{Higgs} \\ & & \downarrow L_{\mathbf{a}} \\ & & \mathbf{Pic}(\tilde{\mathcal{C}}/(B - \Delta)) \otimes \Lambda^\vee \\ & \searrow \scriptstyle \text{aj} \times \text{id} & \parallel \\ & & \mathbf{Bun}_T \end{array}$$

where $\text{aj} : \tilde{\mathcal{C}} \rightarrow \mathbf{Pic}(\tilde{\mathcal{C}}/(B - \Delta))$ denotes the classical Abel-Jacobi map, sending a point $x \in \tilde{C}$ to the line bundle $\mathcal{O}_{\tilde{C}}(x)$ of degree one on \tilde{C} .

We are now ready to construct the line bundles $\mathcal{L}_{x,\lambda}$ on ${}^L\mathbf{Higgs}$ given by $x \in \tilde{C}$, $\lambda \in \Lambda$. For this we will use the well known fact that the Picard gerbe on any smooth family of

curves is self-dual. Note that this is precisely our Theorem B in the abelian case $G = \mathbb{C}^\times$. More precisely, for any smooth compact complex curve Σ , there exists a Poincare sheaf on $\mathcal{P}\mathbf{ic}(\Sigma) \times \mathcal{P}\mathbf{ic}(\Sigma)$ which induces a canonical isomorphism $(\mathcal{P}\mathbf{ic}(\Sigma))^D = \mathcal{P}\mathbf{ic}(\Sigma)$. In fact this isomorphism is induced by the classical Abel-Jacobi map $\mathbf{aj} : \Sigma \rightarrow \mathcal{P}\mathbf{ic}(\Sigma)$:

$$\mathbf{aj}^* : (\mathcal{P}\mathbf{ic}(\Sigma))^D \rightarrow \mathcal{P}\mathbf{ic}(\Sigma).$$

We apply this to the curve \tilde{C} and cross with Λ^\vee to get an induced isomorphism

$$(\mathbf{aj} \times \mathrm{id})^* : (\mathcal{P}\mathbf{ic}(\tilde{C}) \times \Lambda^\vee)^D \xrightarrow{\cong} \mathcal{B}\mathbf{un}_T(\tilde{C}).$$

In particular, for every $x \in \tilde{C}$ and any $\lambda \in \Lambda$ we can find a canonical translation invariant line bundle $\mathbb{L}_{x,\lambda}$ on $\mathcal{P}\mathbf{ic}(\tilde{C}) \times \Lambda^\vee$ such that $(\mathbf{aj} \times \mathrm{id})^* \mathbb{L}_{x,\lambda} = S_{x,\lambda}$. To finish the proof we set

$$\mathfrak{L}_{x,\lambda} := \mathfrak{a}^* \mathbb{L}_{x,\lambda}$$

and invoke the commutative diagram (6) to get the desired identity

$$(\mathbf{aj}^{L_G})^* \mathfrak{L}_{x,\lambda} = (\mathbf{aj}^{L_G})^* \mathfrak{a}^* \mathbb{L}_{x,\lambda} = (\mathbf{aj} \times \mathrm{id})^* \mathbb{L}_{x,\lambda} = S_{x,\lambda}.$$

This concludes the proof of the theorem. \square

The previous duality result extends readily to general reductive groups.

Corollary 3.3 *Let \mathbb{G} be a connected complex reductive group, let ${}^L\mathbb{G}$ be the Langlands dual reductive group, and let C be a smooth compact complex curve. Write $\mathcal{H}\mathbf{iggs}_{\mathbb{G}}$ and $\mathcal{H}\mathbf{iggs}_{({}^L\mathbb{G})}$ for the stacks of K_C -valued Higgs bundles on C with structure group \mathbb{G} and ${}^L\mathbb{G}$ respectively. Then there is an isomorphism $\mathbf{l} : B \xrightarrow{\sim} {}^L B$ of the respective Hitchin bases which gives an identification $B - \Delta \cong {}^L B - {}^L \Delta$. Under this identification one has a canonical isomorphism*

$$(\mathcal{H}\mathbf{iggs}_{({}^L\mathbb{G})})^D \cong \mathcal{H}\mathbf{iggs}_{\mathbb{G}}$$

of commutative group stacks over $B - \Delta$.

Proof. Since \mathbb{G} is connected and reductive, we can always include \mathbb{G} in a short exact sequence

$$(7) \quad 1 \longrightarrow K \longrightarrow G \times H \longrightarrow \mathbb{G} \longrightarrow 1,$$

where K is a finite abelian group, $G = \prod_{i=1}^a G_i$ is a product of complex simple groups, and $H \cong (\mathbb{C}^\times)^b$ is an affine complex torus. Passing to Langlands duals gives the sequence

$$(8) \quad 1 \longrightarrow K^\wedge \longrightarrow {}^L\mathbb{G} \longrightarrow {}^L G \times {}^L H \longrightarrow 1.$$

Next observe that the construction of the Hitchin base, the formation of the moduli stack of Higgs bundles, the definition of the sheaf \mathcal{T} , as well as the operations ${}^L(\bullet)$ and $(\bullet)^D$, all

respect the operation of taking products of groups. Combined with Theorems A and B, and with the standard selfduality of \mathbf{Pic} of a smooth curve, we get an identification of the Hitchin bases for $G \times H$ and ${}^L G \times {}^L H$, as well as a global duality

$$(9) \quad (\mathcal{Higgs}_{L_G \times L_H})^D \cong \mathcal{Higgs}_{G \times H}.$$

Also, since the Hitchin base depends only on the Lie algebra, and not on the Lie group, it follows that the identification of the Hitchin bases for $G \times H$ and ${}^L G \times {}^L H$ can be interpreted as the desired isomorphism $B \cong {}^L B$.

Furthermore, the definition of \mathcal{T} gives short exact sequences of abelian sheaves on $B \times C$:

$$\begin{aligned} 0 &\longrightarrow K \longrightarrow \mathcal{T}_G \oplus H \longrightarrow \mathcal{T}_{\mathbb{G}} \longrightarrow 0 \\ 0 &\longrightarrow K^\wedge \longrightarrow \mathcal{T}_{L_{\mathbb{G}}} \longrightarrow \mathcal{T}_{L_G} \oplus {}^L H \longrightarrow 0. \end{aligned}$$

Applying $R\pi_*[1]$ to the first sequence we get a distinguished triangle in $D^b(B)$:

$$(10) \quad R\pi_* K[1] \xrightarrow{g} R\pi_* \mathcal{T}_G[1] \oplus R\pi_* H[1] \longrightarrow R\pi_* \mathcal{T}_{\mathbb{G}}[1] \longrightarrow R\pi_* K[2].$$

Applying $\underline{R}\mathrm{Hom}(R\pi_*(\bullet), \mathcal{O}^\times)$ to the second sequence, and taking into account the isomorphism (9), we get another triangle

$$(11) \quad \underline{R}\mathrm{Hom}(R\pi_* K^\wedge[1], \mathcal{O}^\times) \xrightarrow{{}^L g} R\pi_* \mathcal{T}_G[1] \oplus R\pi_* H[1] \longrightarrow (R\pi_* \mathcal{T}_{L_{\mathbb{G}}}[1])^D \longrightarrow \underline{R}\mathrm{Hom}(R\pi_* K^\wedge, \mathcal{O}^\times).$$

Now Poincaré duality on a smooth curve \tilde{C} identifies the cohomology of \tilde{C} with coefficients in K with the Pontryagin dual of the cohomology of \tilde{C} with coefficients in K^\wedge . In particular, it induces an isomorphism of complexes

$$\underline{R}\mathrm{Hom}(R\pi_* K^\wedge[1], \mathcal{O}^\times) \cong R\pi_* K[1]$$

which clearly intertwines the maps g and ${}^L g$. In particular the triangles (10) and (11) are isomorphic and so we have a quasi-isomorphism

$$(R\pi_* \mathcal{T}_{L_{\mathbb{G}}}[1])^D \cong R\pi_* \mathcal{T}_{\mathbb{G}}[1].$$

Passing to the associated stacks we obtain the statement of the corollary.

The above argument can also be interpreted geometrically. It shows that the stacks $(\mathcal{Higgs}_{L_{\mathbb{G}}})^D$ and $\mathcal{Higgs}_{\mathbb{G}}$ can both be realized as the quotient of the stack (9) by the commutative group stack of K -torsors on $\tilde{\mathcal{C}}$. \square

The duality isomorphisms in Theorem B and Corollary 3.3 respect all the additional structures on the stacks of Higgs bundles. For instance they respect weight filtrations:

Remark 3.4 The isomorphism of group stacks $({}^L\mathcal{Higgs})^D \cong \mathcal{Higgs}$ in the Theorem B compatible with the isomorphism of abelian schemes

$$({}^L\mathbf{Higgs}_0)^D = \widehat{{}^L P} \cong P = \mathbf{Higgs}_0,$$

constructed in the proof of Theorem A(2). More precisely, if we use \mathbf{l} to identify $B - \Delta$ with ${}^L B - {}^L \Delta$, then the Hitchin fibrations allow us to view \mathcal{Higgs} and ${}^L\mathcal{Higgs}$ as Beilinson 1-motives over $B - \Delta$. This means that \mathcal{Higgs} and ${}^L\mathcal{Higgs}$ are commutative group stacks over $B - \Delta$, which are naturally filtered, with graded pieces which are either abelian varieties, or finite abelian groups, or classifying stacks of finite abelian groups. The filtrations are given as

$$\begin{array}{ccccc} W_0 \mathcal{Higgs} & \supset & W_{-1} \mathcal{Higgs} & \supset & W_{-2} \mathcal{Higgs} \supset 0 \\ \parallel & & \parallel & & \parallel \\ \mathcal{Higgs} & & \mathcal{Higgs}_0 & & BZ(G) \end{array}$$

respectively

$$\begin{array}{ccccc} W_0({}^L\mathcal{Higgs}) & \supset & W_{-1}({}^L\mathcal{Higgs}) & \supset & W_{-2}({}^L\mathcal{Higgs}) \supset 0 \\ \parallel & & \parallel & & \parallel \\ {}^L\mathcal{Higgs} & & {}^L\mathcal{Higgs}_0 & & BZ({}^L G) \end{array}$$

and the duality operation $(\bullet)^D := \underline{\mathrm{Hom}}_{\mathrm{gp}}(\bullet, \mathcal{O}_B^\times[1])$, is compatible with these filtrations. In particular $(\bullet)^D$ transforms each filtered commutative group stack into a stack of the same type, and the isomorphism $({}^L\mathcal{Higgs})^D \cong \mathcal{Higgs}$ respects the filtrations.

Concretely, the above filtrations give rise to short exact sequences of commutative group stacks over $B - \Delta$:

$$(*) \quad \begin{bmatrix} 0 \longrightarrow \mathcal{Higgs}_0 \longrightarrow \mathcal{Higgs} \longrightarrow \pi_1(G) \longrightarrow 0 \\ 0 \longrightarrow BZ(G) \longrightarrow \mathcal{Higgs}_0 \longrightarrow \mathbf{Higgs}_0 \longrightarrow 0 \end{bmatrix}$$

and

$$(**) \quad \begin{bmatrix} 0 \longrightarrow \mathbf{Higgs}_0 \longrightarrow \mathbf{Higgs} \longrightarrow \pi_1(G) \longrightarrow 0 \\ 0 \longrightarrow BZ(G) \longrightarrow \mathcal{Higgs} \longrightarrow \mathbf{Higgs} \longrightarrow 0. \end{bmatrix}$$

Writing the same sequences for ${}^L\mathcal{Higgs}$ and applying $(\bullet)^D$ we get

$$({}^{L*}{}^D) \quad \begin{bmatrix} 0 \longrightarrow ({}^L\mathbf{Higgs}_0)^D \longrightarrow ({}^L\mathcal{Higgs}_0)^D \longrightarrow Z({}^L G)^\wedge \longrightarrow 0 \\ 0 \longrightarrow B\pi_1({}^L G)^\wedge \longrightarrow ({}^L\mathcal{Higgs})^D \longrightarrow ({}^L\mathcal{Higgs}_0)^D \longrightarrow 0 \end{bmatrix}$$

and

$$({}^{L**}{}^D) \quad \begin{bmatrix} 0 \longrightarrow ({}^L\mathbf{Higgs})^D \longrightarrow ({}^L\mathcal{Higgs})^D \longrightarrow Z({}^L G)^\wedge \longrightarrow 0 \\ 0 \longrightarrow B\pi_1({}^L G)^\wedge \longrightarrow ({}^L\mathbf{Higgs})^D \longrightarrow ({}^L\mathbf{Higgs}_0)^D \longrightarrow 0. \end{bmatrix}$$

The fact that the isomorphism (3) respects the filtrations is equivalent to showing that (3) induces an identification of short exact sequences $(*) \cong ({}^{L**}{}^D)$ (equivalently $(**) \cong ({}^{L*}{}^D)$).

Indeed, in the process of proving Theorem B we got compatible isomorphisms of commutative group stacks (or spaces):

$$\begin{array}{ccc}
\mathcal{Higgs} & \cong & ({}^L\mathcal{Higgs})^D \\
\uparrow & & \uparrow \\
\mathcal{Higgs}_0 & \cong & ({}^L\mathbf{Higgs})^D \\
\downarrow & & \downarrow \\
\mathbf{Higgs}_0 & \cong & ({}^L\mathbf{Higgs}_0)^D
\end{array}$$

where in the top row we have the isomorphism (3) from Theorem B, and in the bottom row we have the isomorphism from Theorem A(2). This shows that the isomorphism (3) identifies the sequences (*) with the sequences $({}^{L**D})$.

Remark 3.5 The proof of Theorem A and the calculation in the proof of Lemma 3.2(i) suggest that the duality of Hitchin systems proven in Theorem B probably admits a refinement in the case when the simple group G is of type B_r or C_r . Indeed, the inclusion of sheaves $\mathcal{T}^\circ \subset \mathcal{T} \subset \overline{\mathcal{T}}$ gives rise to three stacky integrable systems over the Hitchin base B :

$$\begin{array}{ccccc}
\mathcal{H}_G^\circ & \longrightarrow & \mathcal{Higgs}_G & \longrightarrow & \overline{\mathcal{H}}_G \\
\parallel & & \parallel & & \parallel \\
\mathcal{Tors}_{\mathcal{T}^\circ} & & \mathcal{Tors}_{\mathcal{T}} & & \mathcal{Tors}_{\overline{\mathcal{T}}}
\end{array}$$

We also have the corresponding coarse moduli spaces, that again admit cohomological interpretation as $H^1(\mathcal{T}_G^\circ)$, $H^1(\mathcal{T}_G)$, and $H^1(\overline{\mathcal{T}}_G)$ respectively. These integrable systems coincide for all groups $G \neq \mathrm{Sp}(r), \mathrm{SO}(2r+1)$, and

$$\begin{aligned}
\mathcal{Higgs}_{\mathrm{Sp}(r)} &\cong \overline{\mathcal{H}}_{\mathrm{Sp}(r)} \\
\mathcal{H}_{\mathrm{SO}(2r+1)}^\circ &\cong \mathcal{Higgs}_{\mathrm{SO}(2r+1)}.
\end{aligned}$$

Now, the calculations in Claim 2.4(ii) and Lemma 3.2(i) give the following values for the stabilizer groups and the groups of connected components of these group stacks:

G	$H^0(\mathcal{T}_G^\circ)$	$H^0(\mathcal{T}_G)$	$H^0(\overline{\mathcal{T}}_G)$
$\mathrm{Sp}(r)$	0	$Z(G) = \mathbb{Z}/2$	$Z(G) = \mathbb{Z}/2$
$\mathrm{SO}(2r+1)$	$Z(G) = 0$	$Z(G) = 0$	$\mathbb{Z}/2$

and

G	$\pi_0(H^1(\mathcal{T}_G^o))$	$\pi_0(H^1(\mathcal{T}_G))$	$\pi_0(H^1(\overline{\mathcal{T}}_G))$
$\mathrm{Sp}(r)$	$\mathbb{Z}/2$	$\pi_1(G) = 0$	$\pi_1(G) = 0$
$\mathrm{SO}(2r+1)$	$\pi_1(G) = \mathbb{Z}/2$	$\pi_1(G) = \mathbb{Z}/2$	0

These values indicate that the duality statement $(\mathbf{Higgs}_{L_G})^D \cong \mathbf{Higgs}_G$ can be augmented to a duality $(\mathcal{H}_{L_G}^o)^D \cong \overline{\mathcal{H}}_G$. In fact, the above tables and the calculation of the cocharacter lattices of the Prym varieties P^o and \overline{P} in Claim 2.5 show that the isomorphism $(\mathcal{H}_{L_G}^o)^D \cong \overline{\mathcal{H}}_G$ holds for the graded pieces with respect to the weight filtrations.

3.4 Hecke eigensheaves

Theorem B has some immediate corollaries. First, we get a categorical equivalence

Corollary 3.6 *Over $B - \Delta$, there is a Fourier-Mukai type equivalence of derived categories*

$$\mathfrak{c} : D_c^b(\mathbf{Higgs}) \xrightarrow{\sim} D_c^b({}^L\mathbf{Higgs}).$$

Moreover, for every $\alpha \in \pi_0(\mathbf{Higgs}) = \pi_1(G) = Z({}^L G)^\wedge$, and every $\beta \in \pi_0({}^L\mathbf{Higgs}) = \pi_1({}^L G) = Z(G)^\wedge$, the functor \mathfrak{c} gives rise to a Fourier-Mukai equivalence

$$D_c^b({}_\beta\mathbf{Higgs}_\alpha) \xrightarrow{\sim} D_c^b({}_\alpha{}^L\mathbf{Higgs}_\beta)$$

for the derived categories of the induced \mathcal{O}^\times -gerbes.

Proof. The isomorphism (3) implies that the \mathcal{O}^\times -gerbes ${}_\alpha{}^L\mathbf{Higgs}_\beta$ and ${}_\beta\mathbf{Higgs}_\alpha$ are compatible, in the sense of [DP03]. In particular, the categorical equivalence statement from [DP03] implies the equivalence of derived categories $D_c^b(\mathbf{Higgs}_0) = D_c^b({}^L\mathbf{Higgs})$. To get the full categorical duality $D_c^b({}^L\mathbf{Higgs}) \cong D_c^b(\mathbf{Higgs})$, one can combine (3) with the duality for representations of commutative group stacks described in Arinkin's appendix to [DP03] (see also [BB06b]), or invoke the recent result [BB06a] of O.Ben-Bassat. In fact, Ben-Bassat's proof works in a much more general context and will imply the full categorical duality even over the discriminant Δ , as long as one can show that the Poincare sheaf on the cameral Pryms extends across Δ . \square

As observed in Remark 3.4, the duality in Theorem B respects the weight filtrations on \mathbf{Higgs} and ${}^L\mathbf{Higgs}$. In particular we have $\mathbf{Higgs}_0 \cong ({}^L\mathbf{Higgs})^D$ and so \mathfrak{c} restricts to a well defined equivalence

$$\mathfrak{c}_0 : D_c^b(\mathbf{Higgs}_0) \xrightarrow{\sim} D_c^b({}^L\mathbf{Higgs}).$$

Finally, we have that the natural orthogonal spanning class of the category $D_c^b(\mathbf{Higgs}_0)$ is transformed by \mathfrak{c} into the class of automorphic sheaves on ${}^L\mathbf{Higgs}$. This is precisely the sense

in which the categorical equivalence \mathfrak{c} can be thought of as a classical limit of the geometric Langlands correspondence. To spell this out, recall that in the proof of Theorem B, we introduced abelianized Hecke correspondences

$$\mathfrak{H}_\mu : {}^L\mathbf{Higgs} \times \tilde{C} \rightarrow {}^L\mathbf{Higgs}$$

labelled by characters $\mu \in \Lambda^\vee = \text{char}(T)$ of T .

Corollary 3.7 *Let (V, φ) be a topologically trivial G -Higgs bundle on C with a cameral cover $p : \tilde{C} \rightarrow C$. The choice of (V, φ) gives:*

- *A ramification twisted W -invariant T -bundle $\mathcal{L}_{(V, \varphi)}$ on \tilde{C} .*
- *A representable structure morphism $\iota : B \text{Aut}((V, \varphi)) \rightarrow \mathbf{Higgs}_0$.*

Write $\mathfrak{o}_{(V, \varphi)} := \iota_* \mathcal{O}_{B \text{Aut}((V, \varphi))}$ for the corresponding sheaf on \mathbf{Higgs}_0 . (This is nothing but the structure sheaf of the stacky point of \mathbf{Higgs}_0 corresponding to (V, φ) .) Then for every $\mu \in \Lambda^\vee$ we have a functorial isomorphism

$$\mathfrak{H}_\mu^* (\mathfrak{c}_0(\mathfrak{o}_{(V, \varphi)})) \cong \mathfrak{c}_0(\mathfrak{o}_{(V, \varphi)}) \boxtimes \mu(\mathcal{L}_{(V, \varphi)}),$$

i.e. $\mathfrak{c}_0(\mathfrak{o}_{(V, \varphi)})$ is a Hecke eigensheaf with eigenvalue $\mathcal{L}_{(V, \varphi)}$.

Proof. This is automatic from the definition of the Hecke correspondences, the abelianization procedure of [DG02, Theorem 6.4], and the fact that the categorical equivalence \mathfrak{c} is compatible with the usual Fourier-Mukai equivalence of \mathbf{Higgs}_0 and ${}^L\mathbf{Higgs}_0$. \square

4 The topological structure of a cameral Prym

In this section we discuss the cohomology groups describing the cocharacter lattices of cameral Prym varieties and the behavior of those groups under Poincare duality. Most of the material here is well known but we couldn't find it in the literature, in the form needed for the proof of Theorem A. We include it here in an attempt to make the paper self-contained. The results in Section 4.2 are new. We give an explicit description of the cocharacter lattice of a cameral Prym in terms of the local monodromies of a cameral cover. We used this result in the proof of Claim 2.4 to analyze the connected components of the Hitchin fiber but is also of independent interest.

4.1 Remarks on local system cohomology

Let C be a smooth compact complex curve of genus g and let $S = \{s_1, \dots, s_b\} \subset C$ be a finite set of points. We write $U := C - S$ for the complement of S and denote by $\iota : S \hookrightarrow C$ and $j : U \hookrightarrow C$ the corresponding closed and open inclusions. We will also fix a base point $\mathfrak{o} \in U$.

Let A be a local system on U of free abelian groups of rank r . Let $A^\vee := \underline{\text{Hom}}_{\mathbb{Z}_U}(A, \mathbb{Z}_U)$ denote the dual local system. We want to understand the cohomology of j_*A in concrete terms and to find the precise relationship between the cohomology of j_*A and $j_*(A^\vee)$. This is all standard for local systems of vector spaces, see e.g. [Loo97], but it requires some care for local system of free abelian groups.

Suppose $s_i \in S$ and let $s_i \in \mathbf{D}_i \subset C$ be a small disc centered at s_i and not containing any other point of S . Fix a point $\mathfrak{o}_i \in \partial \mathbf{D}_i$ and let c_i denote the loop starting and ending at \mathfrak{o}_i and traversing $\partial \mathbf{D}_i$ once in the positive direction. Write $\text{mon}(c_i) : A_{\mathfrak{o}_i} \rightarrow A_{\mathfrak{o}_i}$ for the monodromy operator associated with c_i . Now, from the definition of the direct image and the fact that A is locally constant we get the following description of the stalk of j_*A at s_i :

$$\begin{aligned} (j_*A)_{s_i} &= \varprojlim \{ H^0(V \cap U, A) \mid s_i \in V, V \subset C \text{ - open} \} \\ &= H^0(\mathbf{D}_i - \{s_i\}, A) \\ &= (A_{\mathfrak{o}_i})^{\text{mon}(c_i)}. \end{aligned}$$

Here, as usual $(A_{\mathfrak{o}_i})^{\text{mon}(c_i)} := \{a \in A_{\mathfrak{o}_i} \mid \text{mon}(c_i)(a) = a\}$ denotes the invariants of the $\text{mon}(c_i)$ -action.

To organize things better, we choose an ordered system of arcs $\{a_i\}_{i=1}^b$ in $C - \cup_{i=1}^b \mathbf{D}_i$ which connect the base point \mathfrak{o} with each of the points \mathfrak{o}_i as in Figure 1.

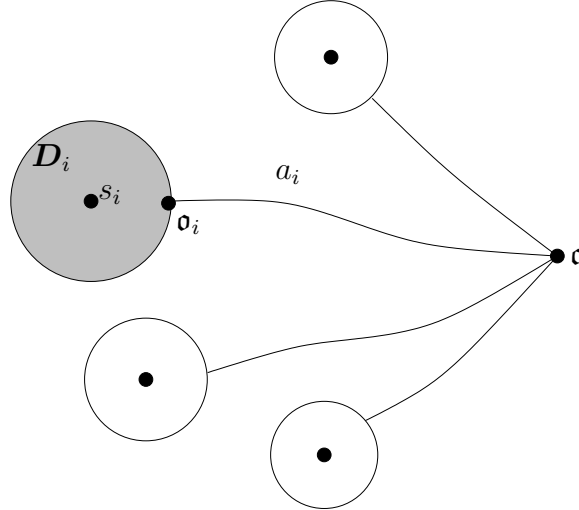


Figure 1: An arc system for $S \subset C$.

A choice of an arc system yields a collection of elements $\gamma_i \in \pi_1(U, \mathfrak{o})$. Geometrically γ_i is the \mathfrak{o} -based loop in U obtained by tracing a_i , followed by tracing c_i and then tracing back a_i in the opposite direction. Since parallel transport along a_i identifies the stalks $A_{\mathfrak{o}}$ and $A_{\mathfrak{o}_i}$ and conjugates the monodromy transformation $\text{mon}(\gamma_i)$ into the monodromy transformation

$\text{mon}(c_i)$, it follows that we also have the identification

$$(j_*A)_{s_i} = (A_{\mathfrak{o}})^{\text{mon}(\gamma_i)}$$

for all $s_i \in S$. To simplify notation we set $\rho_i := \text{mon}(\gamma_i)$.

With this notation in place we are ready to analyze the cohomology of the sheaves A and j_*A . First note that U is a smooth 2-manifold and so has homological dimension 2 with respect to compactly supported cohomology [Ive86, Section III.9] or [Dim04, Section 3.1]. Also, since A is a locally constant sheaf, it can not have any compactly supported sections and so the only compactly supported cohomology groups of A that can be potentially non-zero are $H_c^1(U, A)$ and $H_c^2(U, A)$. On the other hand, since A is a local system, its cohomology is homotopy invariant. Taking into account the fact that U is homotopy equivalent to a bouquet of circles, we conclude that the only cohomology groups of A that are potentially non-zero are $H^0(U, A)$ and $H^1(U, A)$. Furthermore the following version of Poincare duality holds for these groups:

Lemma 4.1 *The cup product pairing*

$$(\text{cup}) \quad H_c^k(U, A) \otimes H^{2-k}(U, A^\vee) \xrightarrow{\cup} H_c^2(U, A \otimes A^\vee) \xrightarrow{\int \text{tr}} \mathbb{Z},$$

induces a perfect pairing between the free abelian groups $H^1(U, A)_{\text{tf}}$ and $H_c^1(U, A^\vee)_{\text{tf}}$. Moreover $H^0(U, A)$ has no torsion and the cup product pairing (cup) induces a perfect pairing between $H^0(U, A)$ and $H_c^2(U, A^\vee)_{\text{tf}}$.

Proof. We will use Verdier duality and the universal coefficients theorem.

Since U is an orientable 2-manifold, the general Verdier duality [Ver95, Ive86, GM03] on U yields an isomorphism in $D(\mathbb{Z} - \text{mod})$:

$$R\text{Hom}_{\mathbb{Z}}(R\Gamma_c(U, A), \mathbb{Z}) \cong R\underline{\text{Hom}}_{\mathbb{Z}_U}(A, \mathbb{Z}_U[2]).$$

In particular we have isomorphisms of cohomology groups

$$H^k(R\text{Hom}_{\mathbb{Z}}(R\Gamma_c(U, A), \mathbb{Z})) \cong \text{Ext}_U^{k+2}(A, \mathbb{Z}_U).$$

Furthermore, we can use the local-to-global spectral sequence

$$H^p(U, \underline{\text{Ext}}_U^q(A, \mathbb{Z}_U)) \Rightarrow \text{Ext}_U^{p+q}(A, \mathbb{Z}_U),$$

to compute $\text{Ext}_U^\bullet(A, \mathbb{Z}_U)$ in terms of cohomology. Namely, since the stalks of A and \mathbb{Z}_U are free abelian groups of finite rank, it follows that $\underline{\text{Ext}}_U^k(A, \mathbb{Z}_U) = 0$ for all $k \geq 1$ and so the local-to-global spectral sequence degenerates at E_2 and yields isomorphisms

$$\text{Ext}_U^k(A, \mathbb{Z}_U) \cong H^k(U, A^\vee)$$

for all $k \geq 0$. In summary we have

$$\begin{aligned} H^0(U, A^\vee) &= H^0(R \operatorname{Hom}_{\mathbb{Z}}(R\Gamma_c(U, A), \mathbb{Z})) \\ H^1(U, A^\vee) &= H^1(R \operatorname{Hom}_{\mathbb{Z}}(R\Gamma_c(U, A), \mathbb{Z})) \\ H^k(U, A^\vee) &= H^k(R \operatorname{Hom}_{\mathbb{Z}}(R\Gamma_c(U, A), \mathbb{Z})) = 0, \text{ for all } k \geq 2, \end{aligned}$$

where the vanishing for $k \geq 2$ follows from the homotopy invariance of cohomology with coefficients in a local system and the fact that U retracts onto a bouquet of circles.

To compute the right hand side for $k = 0$ and 1 we can either use the universal coefficients theorem [God73, Theorem 5.4.2], [Dim04, Theorem 1.4.5], or argue directly as follows. Consider the complex of abelian groups

$$R^\bullet := R \operatorname{Hom}_{\mathbb{Z}}(R\Gamma_c(U, A), \mathbb{Z}).$$

The universal coefficients spectral sequence is the spectral sequence associated with the stupid filtration on this complex. In this case it reads

$$E_2^{pq} = \operatorname{Ext}^p(H_c^q(U, A), \mathbb{Z}) \Rightarrow H^{p-q}(R^\bullet),$$

or in combination with Verdier duality

$$E_2^{pq} = \operatorname{Ext}^p(H_c^q(U, A), \mathbb{Z}) \Rightarrow H^{2+p-q}(U, A^\vee).$$

Since \mathbb{Z} has injective dimension one, this sequence degenerates at E_2 and we get short exact sequences

$$\begin{aligned} (12) \quad 0 &\longrightarrow \operatorname{Ext}^1(H_c^2(U, A), \mathbb{Z}) \longrightarrow H^1(U, A^\vee) \xrightarrow{(\dagger)} H_c^1(U, A)^\vee \longrightarrow 0 \\ 0 &\longrightarrow \operatorname{Ext}^1(H_c^3(U, A), \mathbb{Z}) \longrightarrow H^0(U, A^\vee) \xrightarrow{(\ddagger)} H_c^2(U, A)^\vee \longrightarrow 0, \end{aligned}$$

where the maps (\dagger) and (\ddagger) are induced from the cup product pairing (\mathbf{cup}) .

Since $\operatorname{Ext}^1(H_c^2(U, A), \mathbb{Z}) = H_c^2(U, A)_{\operatorname{tors}}^\wedge$, it follows that the torsion subgroup $H^1(U, A^\vee)_{\operatorname{tors}}$ of $H^1(U, A^\vee)$ is equal to $H_c^2(U, A)_{\operatorname{tors}}^\wedge$, and hence the map (\dagger) induces an isomorphism

$$H^1(U, A^\vee)_{\operatorname{tf}} \rightarrow H_c^1(U, A)^\vee.$$

Taking into account the fact that $H_c^1(U, A)^\vee = (H_c^1(U, A)_{\operatorname{tf}})^\vee$ we conclude that (\mathbf{cup}) induces an isomorphism between the free abelian groups $H^1(U, A^\vee)_{\operatorname{tf}}$ and $H_c^1(U, A)_{\operatorname{tf}}^\vee$. \square

Next we will use this information to compute the cohomology of A and j_*A explicitly in terms of the monodromy. We begin with a standard lemma:

Lemma 4.2 *The direct image sheaves $R^k j_* A$ can be described as follows:*

(a) The sheaf j_*A fits in a short exact sequence

$$0 \rightarrow j_!A \rightarrow j_*A \rightarrow \bigoplus_{i=1}^b A_{\mathfrak{o}}^{\rho_i} \rightarrow 0,$$

where $j_!$ denotes the pushforward with compact supports.

(b) The sheaf $R^1 j_*A$ satisfies

$$R^1 j_*A = \bigoplus_{i=1}^b (A_{\mathfrak{o}})_{\rho_i},$$

where $(A_{\mathfrak{o}})_{\rho_i} := A_{\mathfrak{o}}/(1 - \rho_i)A_{\mathfrak{o}}$ denotes the group of coinvariants of the ρ_i -action on $A_{\mathfrak{o}}$.

(c) $R^k j_*A = 0$ for all $k \geq 2$.

Proof. The usual gluing [BBD82, Section 1.4], [GM03, Chapter 4, Exercise 3] for the inclusions $U \xrightarrow{j} C \xleftarrow{i} S$ yields a distinguished triangle

$$j_!j^{-1}F^\bullet \rightarrow F^\bullet \rightarrow i_*i^{-1}F^\bullet \rightarrow j_!F^\bullet[1]$$

defined for every $F^\bullet \in D(\mathbb{Z}_U - \text{mod})$.

Now take $F^\bullet := Rj_*A$. Since the standard adjunction map $j^{-1}Rj_*A \rightarrow A$ is an isomorphism, we get a distinguished triangle

$$j_!A \rightarrow Rj_*A \rightarrow i_*i^{-1}Rj_*A \rightarrow j_!A[1].$$

The induced long exact sequence of cohomology sheaves reads:

$$\begin{array}{ccccccc} 0 & \longrightarrow & j_!A & \longrightarrow & j_*A & \longrightarrow & i_*i^{-1}j_*A \\ & & & & & & \searrow \\ & & & & & & 0 \longrightarrow R^1j_*A \longrightarrow i_*i^{-1}R^1j_*A \\ & & & & & & \searrow \\ & & & & & & 0 \longrightarrow R^2j_*A \longrightarrow i_*i^{-1}R^2j_*A \\ & & & & & & \searrow \\ & & & & & & 0 \longrightarrow \dots \end{array}$$

This proves (a) and shows that

$$R^k j_*A = \bigoplus_{i=1}^b (R^k j_*A)_{s_i}$$

for all $k \geq 1$. On the other hand we have

$$\begin{aligned} (R^k j_*A)_{s_i} &= \varprojlim \{ H^k(V \cap U, A) \mid s_i \in V, V \subset C - \text{open} \} \\ &= H^k(\mathbf{D}_i - \{s_i\}, A) \\ &= H^k(c_i, A). \end{aligned}$$

The cohomology $H^k(c_i, A)$ can be computed either as group cohomology or as Čech cohomology. In either interpretation, note that for $k \geq 2$ we have $H^k(c_i, A) = 0$ for reasons of homological dimension. This proves **(c)**.

To prove **(b)** we will choose an open covering for c_i which is acyclic for A . We cover c_i by two overlapping intervals I_1 and I_2 and we denote the two connected components of the intersection $I_1 \cap I_2$ by I_{12}^1 and I_{12}^2 . Without a loss of generality we may assume that the covering $\{I_1, I_2\}$ is chosen so that $\mathfrak{o}_i \in I_{12}^2$. The covering $\{I_1, I_2\}$ is depicted on Figure 2:

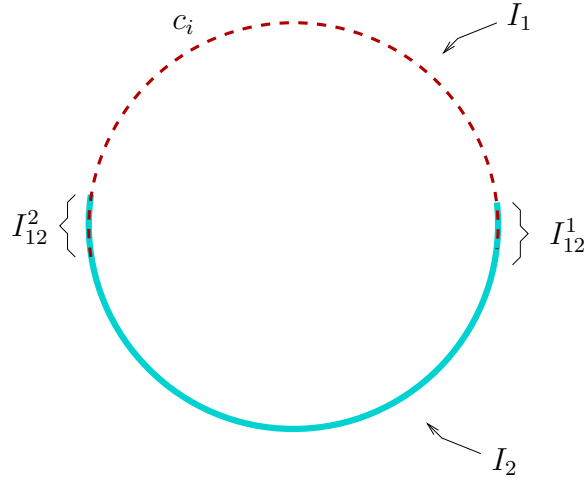


Figure 2: The open cover $\{I_1, I_2\}$ of c_i .

Since the intervals I_1 , I_2 , I_{12}^1 , and I_{12}^2 are contractible, it follows that A is a constant sheaf when restricted to any of these intervals. Consider the group $\mathfrak{A} := A_{\mathfrak{o}_i}$ and for any space Z write \mathfrak{A}_Z for the corresponding constant sheaf on Z . Now choose trivializations

$$\begin{aligned} h_1 : A|_{I_1} &\xrightarrow{\sim} \mathfrak{A}_{I_1}, \\ h_2 : A|_{I_2} &\xrightarrow{\sim} \mathfrak{A}_{I_2}, \end{aligned}$$

and normalize the choice so that the maps

$$\begin{aligned} h_{1|I_{12}^1} : A|_{I_{12}^1} &\xrightarrow{\sim} \mathfrak{A}_{I_{12}^1}, \\ h_{2|I_{12}^1} : A|_{I_{12}^1} &\xrightarrow{\sim} \mathfrak{A}_{I_{12}^1} \end{aligned}$$

coincide. (Note that this is always possible since every locally defined automorphism of the constant sheaf extends to a globally defined automorphism.) Now the gluing map

$$h_1 \circ h_2^{-1} : \mathfrak{A}_{I_{12}^2} \rightarrow \mathfrak{A}_{I_{12}^1}$$

is given explicitly as

$$\begin{array}{ccc} \mathfrak{A}_{I_{12}^2} & \xrightarrow{h_1 \circ h_2^{-1}} & \mathfrak{A}_{I_{12}^1} \\ \parallel & & \parallel \\ \mathfrak{A}_{I_{12}^2} \oplus \mathfrak{A}_{I_{12}^1} & \xrightarrow{\text{id} \oplus \tau} & \mathfrak{A}_{I_{12}^2} \oplus \mathfrak{A}_{I_{12}^1} \end{array}$$

where $\mathfrak{r} := \text{Mon}(c_i)$ is the monodromy of A for going once along c_i in the counterclockwise direction.

Now the Čech complex computing the cohomology of A on c_i can be identified explicitly as

$$\begin{array}{ccc}
\check{C}^0(A) & \xrightarrow{\delta} & \check{C}^1(A) \\
\parallel & & \parallel \\
\Gamma(I_1, A) \oplus \Gamma(I_2, A) & \xrightarrow{(s,t) \mapsto t-s} & \Gamma(I_{12}, A) \\
\downarrow h_1 \oplus h_2 & & \downarrow h_1 \oplus h_1 \\
\mathfrak{A} \oplus \mathfrak{A} & \xrightarrow{(a,b) \mapsto (b-a, \mathfrak{r}(b)-a)} & \mathfrak{A} \oplus \mathfrak{A}
\end{array}$$

In particular we get

$$H^1(c_i, A) = \mathfrak{A}_{\mathfrak{r}} = \mathfrak{A}/(\mathfrak{r} - 1)\mathfrak{A},$$

which proves (b). \square

Remark 4.3 One can easily compute $H^k(c_i, A)$ via group cohomology. Since $c_i \cong S^1 = K(\mathbb{Z}, 1)$ we have

$$H^k(c_i, A) = H^k(\pi, M),$$

where $\pi := \pi_1(c_i, \mathfrak{o}_i) \cong \mathbb{Z}$, and M denotes the π module $(\mathfrak{A}, \mathfrak{r})$. Now the group ring $\mathbb{Z}[\pi] = \mathbb{Z}[t]$ is a polynomial ring in one variable over \mathbb{Z} and so \mathbb{Z} has a two step resolution

$$0 \longrightarrow \mathbb{Z}[t] \xrightarrow{\partial} \mathbb{Z}[t] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

by free $\mathbb{Z}[\pi]$ -modules. Here $\varepsilon : \mathbb{Z}[t] \rightarrow \mathbb{Z}$ is the augmentation map $\varepsilon(p(t)) := p(0)$, and $\partial : \mathbb{Z}[t] \rightarrow \mathbb{Z}[t]$ is the map $\partial p(t) := (t-1)p(t)$ of multiplication by $t-1$. In particular, for any π module $M = (\mathfrak{A}, \mathfrak{r})$ we can compute $H^\bullet(\pi, M)$ as the cohomology of the complex

$$\begin{array}{ccc}
(\text{degree } 0) & & (\text{degree } 1) \\
\text{Hom}_\pi(\mathbb{Z}[t], M) & \xrightarrow{\delta} & \text{Hom}_\pi(\mathbb{Z}[t], M) \\
\parallel & & \parallel \\
\mathfrak{A} & \xrightarrow{\mathfrak{r}-1} & \mathfrak{A}.
\end{array}$$

In other words $H^0(\pi, M) = \mathfrak{A}^{\mathfrak{r}}$, $H^1(\pi, M) = \mathfrak{A}_{\mathfrak{r}}$, and $H^k(\pi, M) = 0$ for $k \geq 2$.

As a consequence of the calculation of $R^k_{j*}A$ one immediately gets the following:

Lemma 4.4 *The cohomology of the sheaf j_*A satisfies:*

$$\begin{aligned} H^0(C, j_*A) &= A_{\mathfrak{o}}^{\pi_1(U, \mathfrak{o})} \\ H^1(C, j_*A) &= \text{im} [H_c^1(U, A) \rightarrow H^1(U, A)] \\ H^2(C, j_*A)_{\text{tf}} &= ((A_{\mathfrak{o}}^{\vee})^{\pi_1(U, \mathfrak{o})})^{\vee} \\ H^2(C, j_*A)_{\text{tor}} &= H^1(U, A^{\vee})_{\text{tor}}^{\wedge}. \end{aligned}$$

Proof. Using the short exact sequence in part (a) of Lemma 4.2 and taking into account the fact that $H^k(C, j_!A) = H_c^k(U, A)$ we get a long exact sequence in cohomology

$$(13) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H_c^0(U, A) & \longrightarrow & H^0(C, j_* A) & \longrightarrow & \bigoplus_{i=1}^b (A_{\mathbf{o}})^{\rho_i} \\ & & & & & & \downarrow \\ & & & & & & H_c^1(U, A) \longrightarrow H^1(C, j_* A) \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & H_c^2(U, A) \longrightarrow H^2(C, j_* A) \longrightarrow 0 \end{array}$$

Note also that U is a two dimensional manifold and so $H_c^k(U, A) = 0$ for all $k \geq 3$. Together with the above long exact sequence this implies that $H^k(C, j_*A) = 0$ for all $k \geq 3$.

Furthermore the natural evaluation map $H^0(C, j_*A) \rightarrow A_{\mathfrak{o}}$ factors as

$$H^0(C, j_*A) \xrightarrow{\sim} H^0(U, A) \hookrightarrow A_0$$

and so

$$(14) \quad H^0(C, j_*A) = (A_{\mathfrak{o}})^{\pi_1(U, \mathfrak{o})}.$$

Also, since A is locally constant, we have $H_c^0(U, A) = 0$ and hence

$$(15) \quad 0 \longrightarrow (A_{\mathfrak{o}})^{\pi_1(U, \mathfrak{o})} \longrightarrow \bigoplus_{i=1}^b (A_{\mathfrak{o}})^{\rho_i} \longrightarrow H_c^1(U, A) \longrightarrow H^1(C, j_* A) \longrightarrow 0,$$

and

$$(16) \quad H^2(C, j_* A) = H_c^2(U, A).$$

Now combining the second sequence in (12) with the vanishing $H_c^3(U, A) = 0$, we see that (cup) induces an isomorphism

$$H_c^2(U, A)^\vee \cong H^0(U, A^\vee) \cong (A_\circ^\vee)^{\pi_1(U, \circ)}.$$

Also, since $\text{Ext}^1(H_c^2(U, A), \mathbb{Z}) = H_c^2(U, A)_{\text{tor}}^\wedge$ is torsion and $H_c^1(U, A)^\vee$ is torsion free, the first sequence in (12) yields $H^1(U, A^\vee)_{\text{tor}} = H_c^2(U, A)_{\text{tor}}^\wedge$.

Finally, by the Leray spectral sequence (which in this case is just Mayer-Vietoris) for $j : U \rightarrow C$ and the sheaf A , we get an exact sequence

$$(17) \quad 0 \longrightarrow H^1(C, j_* A) \longrightarrow H^1(U, A) \longrightarrow H^0(C, R^1 j_* A) \longrightarrow H^2(C, j_* A) \rightarrow 0.$$

The composition $H_c^1(U, A) \rightarrow H^1(C, j_* A) \hookrightarrow H^1(U, A)$ is the natural map from compactly supported to ordinary cohomology and so $H^1(C, j_* A) = \text{im}[H_c^1(U, A) \rightarrow H^1(U, A)]$. \square

Corollary 4.5 *For any local system A of finite rank free abelian groups on U we have a natural identification*

$$H^1(C, j_* A^\vee)_{\text{tf}} = \text{im}[H^1(U, A)^\vee \rightarrow H^1(C, j_* A)^\vee]$$

Proof. By the previous lemma we have an identification

$$H^1(C, j_* A^\vee) = \text{im}[H_c^1(U, A^\vee) \rightarrow H^1(U, A^\vee)].$$

In particular we have $H^1(C, j_* A^\vee)_{\text{tf}} = \text{im}[H_c^1(U, A^\vee)_{\text{tf}} \rightarrow H^1(U, A^\vee)_{\text{tf}}]$. On the other hand, by Lemma 4.1 the natural map $H_c^1(U, A^\vee)_{\text{tf}} \rightarrow H^1(U, A^\vee)_{\text{tf}}$ is equal to minus the transpose of the map $H_c^1(U, A)_{\text{tf}} \rightarrow H^1(U, A)_{\text{tf}}$. Since by Lemma 4.4 we have

$$H^1(C, j_* A) = \text{im}[H_c^1(U, A) \rightarrow H^1(U, A)],$$

we get that

$$\text{im}[H^1(U, A)^\vee \rightarrow H_c^1(U, A)^\vee] = \text{im}[H^1(U, A)^\vee \rightarrow H^1(C, j_* A)^\vee],$$

which yields the lemma. \square

4.2 The cocharacters of a cameral Prym

Let $p : \tilde{C} \rightarrow C$ be a generic Galois cameral cover as in the proof of Theorem A. Let $\{s_1, \dots, s_b\} \subset C$ be the branch points of this cover. We write $j : U \hookrightarrow C$ for the inclusion of the complement, and $p^\circ : p^{-1}(U) \rightarrow U$ for the unramified part of p . Define a local system A on U by $A := (p^\circ^* \Lambda)^W$. The canonical identification ${}^L \Lambda = \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z})$ gives also an identification ${}^L A = A^\vee = \underline{\text{Hom}}(\Lambda, \mathbb{Z})$.

Fix a base point $\mathfrak{o} \in U$ and choose an arc system as in the previous section. We choose once and for all an identification $A_\mathfrak{o} \cong \Lambda$. By definition, the monodromy $\text{mon}_\mathfrak{o} : \pi_1(U, \mathfrak{o}) \rightarrow GL(\Lambda)$ factors through $W \subset SO(\Lambda, \langle \bullet, \bullet \rangle) \subset GL(\Lambda)$. By the genericity assumption on $p : \tilde{C} \rightarrow C$, it follows that the monodromy image of each generator $\gamma_i \in \pi_1(U, \mathfrak{o})$ is a reflection $\rho_i : \Lambda \rightarrow \Lambda$ corresponding to some root α_i of \mathfrak{g} . Explicitly we have $\rho_i(\lambda) = \lambda - (\alpha_i, \lambda) \cdot \alpha_i^\vee$, where $\alpha_i^\vee \in \Lambda$ is the coroot corresponding to α_i . Since for each root α the divisor $D^\alpha \subset \text{tot}(K_C \otimes \mathfrak{t})$ is ample, it follows that the collection of roots $\{\alpha_1, \dots, \alpha_b\}$ contains both long and short roots of \mathfrak{g} . Let $\varepsilon_i := \varepsilon_{\alpha_i, G}$ and $\varepsilon_i^\vee := \varepsilon_{\alpha_i^\vee, G}$.

In the proof of Theorem A we described the cocharacter lattice of the cameral Prym P corresponding to $p : \tilde{C} \rightarrow C$ and G in terms of the first cohomology of the sheaves j_*A and j_*A^\vee on C . We now give explicit formulas for these cohomology groups. Note that without a loss of generality we may assume that S , the arcs a_i and the discs \mathbf{D}_i are all contained in the interior of a disc $\mathbf{D} \subset C$ for which $\mathfrak{o} \in \partial\mathbf{D}$. In particular we can choose a collection of \mathfrak{o} based loops $\delta_1, \delta_2, \dots, \delta_{2g} \subset C - \mathbf{D}$ which intersect only at \mathfrak{o} and form a system of standard a - b generators for the fundamental group $\pi_1(C, \mathfrak{o})$ of the compact curve C . Choosing the orientation of the loops δ_j and γ_i appropriately we get a presentation of the fundamental group of U :

$$\pi_1(U, \mathfrak{o}) = \left\langle \delta_1, \dots, \delta_{2g}, \gamma_1, \dots, \gamma_b \mid \prod_{j=1}^g [\delta_j, \delta_{g+j}] \prod_{i=1}^b \gamma_i = 1 \right\rangle$$

To simplify notation we set $\mathbf{w}_i := \text{mon}_{\mathfrak{o}}(\delta_i) \in W$. Finally, it will be convenient to add to S an extra point $s_0 \neq \mathfrak{o} \in U$ and a loop γ_0 around s_0 , oriented so that the relation defining $\pi_1(U - \{s_0\}, \mathfrak{o})$ is $\gamma_0 = \prod_{j=1}^g [\delta_j, \delta_{g+j}] \prod_{i=1}^b \gamma_i$. Note that we have $\rho_0 = \text{mon}_{\mathfrak{o}}(\gamma_0) = 1 \in W$. Note also that the deletion of s_0 from U will not affect our interpretation of $H^1(C, j_*A)$ as a kernel of a homomorphism. That is, we still have

$$\begin{aligned} H^1(C, j_*A) &= \ker [H^1(U, A) \rightarrow \oplus_{i=1}^b \Lambda / (1 - \rho_i) \Lambda] \\ &= \ker [H^1(U - \{s_0\}, A) \rightarrow \oplus_{i=0}^b \Lambda / (1 - \rho_i) \Lambda], \end{aligned}$$

since the deletion adds a copy of Λ to both $H^1(\bullet, A)$ and $R^1 j_*A$.

Proposition 4.6 (a) *There is a natural isomorphism*

$$H^1(U - \{s_0\}, A) \cong \frac{\Lambda^{2g+b}}{(1 - \mathbf{w}_1, \dots, 1 - \mathbf{w}_{2g}, 1 - \rho_1, \dots, 1 - \rho_b) \Lambda}$$

which depends only on the choice of an arc system, the $a - b$ loops δ_j , and the identification $A_{\mathfrak{o}} \cong \Lambda$. In addition, there is a non-canonical isomorphism

$$H^1(U - \{s_0\}, A) = H^1(C, \Lambda) \oplus \frac{\Lambda^b}{(1 - \rho_1, \dots, 1 - \rho_b) \Lambda}.$$

(b) *Under the isomorphism $H^1(U - \{s_0\}, A) = H^1(C, \Lambda) \oplus (\Lambda^b / (1 - \rho_1, \dots, 1 - \rho_b) \Lambda)$, the subgroup $H^1(C, j_*A) \subset H^1(U, A)$ can be identified as*

$$H^1(C, j_*A) = H^1(C, \Lambda) \oplus \frac{\ker \left[\sum_{i=1}^b \prod_{k=1}^{i-1} \rho_k : \oplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee \rightarrow \Lambda \right]}{(1 - \rho_1, \dots, 1 - \rho_b) \Lambda}$$

Proof. (a) The surface $U - \{s_0\} = C - \{s_0, s_1, \dots, s_b\}$ is homotopy equivalent to the bouquet consisting of the $2g + b$ oriented circles $\delta_1, \dots, \delta_{2g}, \gamma_1, \dots, \gamma_b$, where all circles are attached to each other at the point \mathfrak{o} . The fundamental group π of this bouquet of circles is a free group

on the generators $\delta_1, \dots, \delta_{2g}, \gamma_1, \dots, \gamma_b$, and the local system A corresponds to the action on of this free group on Λ specified by the monodromy transformations $(\mathbf{w}_1, \dots, \mathbf{w}_{2g}, \rho_1, \dots, \rho_b) \in W^{2g+b}$. As in Remark 4.3, the trivial π -module \mathbb{Z} has a free $\mathbb{Z}[\pi]$ -resolution given by

$$0 \longrightarrow \left(\bigoplus_{j=1}^{2g} \mathbb{Z}[\pi] e_{\delta_j} \right) \oplus \left(\bigoplus_{i=1}^b \mathbb{Z}[\pi] e_{\gamma_i} \right) \xrightarrow{\partial} \mathbb{Z}[\pi] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $\left(\bigoplus_{j=1}^{2g} \mathbb{Z}[\pi] e_{\delta_j} \right) \oplus \left(\bigoplus_{i=1}^b \mathbb{Z}[\pi] e_{\gamma_i} \right)$ is the free $\mathbb{Z}[\pi]$ module on generators $e_{\delta_j}, e_{\gamma_i}$ and $\partial e_{\delta_j} = 1 - \delta_j, \partial e_{\gamma_i} = 1 - \gamma_i$.

Applying $\text{Hom}_{\mathbb{Z}[\pi]}(\bullet, \Lambda)$ and computing cohomology we get the identification

$$H^1(U - \{s_0\}, A) = H^1(\pi, \Lambda) \cong \frac{\Lambda^{2g+b}}{(1 - \mathbf{w}_1, \dots, 1 - \mathbf{w}_{2g}, 1 - \rho_1, \dots, 1 - \rho_b) \Lambda}.$$

Next we claim that by making appropriate choices, the topological description of the cameral cover can be brought into a particularly simple form. Recall [DDP05], that there is a natural inclusion

$$(18) \quad (H^0(C, K_C) \otimes \mathfrak{t})/W \hookrightarrow B.$$

A cameral cover $p_b : \tilde{C}_b \rightarrow C$ corresponding to a generic point b in the image of (18) is reduced but completely reducible:

$$\tilde{C}_b = \bigcup_{w \in W} \tilde{C}_{b,w},$$

with each irreducible component $\tilde{C}_{b,w}$ isomorphic to C . Let $\mathbf{D} \subset C$ be a disc containing the image of all the singular points (= intersection of components) of \tilde{C}_b . We get that $p^{-1}(C - \mathbf{D})$ is completely disconnected:

$$(19) \quad p^{-1}(C - \mathbf{D}) = \coprod_{w \in W} [C - \mathbf{D}]_w, \quad [C - \mathbf{D}]_w := \tilde{C}_{b,w} \cap p^{-1}(C - \mathbf{D}),$$

with each connected components $[C - \mathbf{D}]_w$ isomorphic to $C - \mathbf{D}$.

A general cameral cover $\tilde{C}_{b'}$ with $b' \in B - \Delta$ near $b \in B$ will be smooth and will still satisfy (19). By taking all the γ_i in \mathbf{D} and all the δ_j in $C - \mathbf{D}$ we have $\mathbf{w}_j = 1, j = 1, \dots, 2g$. Consequently

$$(20) \quad H^1(U - \{s_0\}, A) = H^1(C, \Lambda) \oplus \frac{\Lambda^b}{(1 - \rho_1, \dots, 1 - \rho_b) \Lambda},$$

for the cover $\tilde{C}_{b'}$. Since $B - \Delta$ is connected, it follows that (20) will hold for any $\tilde{C}_{b''}$, $b'' \in B - \Delta$ and an appropriate choice of γ_i 's and δ_j 's.

(b) As we argued in the previous section, the group $H^1(C, j_* A)$ is the kernel of the natural map $H^1(U - \{s_0\}, A) \rightarrow H^0(C, R^1 j_* A) = \bigoplus_{i=0}^b \Lambda / (1 - \rho_i) \Lambda$. Under the identification (20), it

is immediate that $H^1(C, \Lambda)$ is contained in the kernel of this map and that the restriction of the map to the summand $\Lambda^b/(1 - \rho_1, \dots, 1 - \rho_b)\Lambda$ is given by

$$(21) \quad (\lambda_1, \dots, \lambda_b) \mapsto (\varphi(\lambda_1, \dots, \lambda_b), \lambda_1 + (1 - \rho_1)\Lambda, \dots, \lambda_b + (1 - \rho_b)\Lambda),$$

where the map $\varphi : \Lambda^b \rightarrow \Lambda/(1 - \rho_0)\Lambda = \Lambda$ corresponds to the relation $\prod_{j=1}^{2g} [\delta_j, \delta_{g+j}] \prod_{i=1}^b \gamma_i = \gamma_0$.

In general, suppose that we are given a bouquet of circles $V = c_1 \vee \dots \vee c_n$ and suppose we have a word $d = \prod_{i=1}^p d_i$ in $\pi := \pi_1(V)$ for which all d_i 's are in $\{c_1, \dots, c_n, c_1^{-1}, \dots, c_n^{-1}\} \subset \pi$. Consider the cyclic subgroup in π generated by d and let M be some π -module. The inclusion $\langle d \rangle \subset \pi$ induces a map on cohomology $H^1(\pi, M) \rightarrow H^1(\langle d \rangle, M)$ which can be explicitly calculated. For this we only need to consider the tree which is the universal cover of V and follow the branches of that tree labeled by the letters d_i in the word d . More invariantly this corresponds to a map of resolutions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}[\langle d \rangle]e_d & \xrightarrow{\partial} & \mathbb{Z}[\langle d \rangle] & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow (**) & & \downarrow (*) & & \parallel \\ 0 & \longrightarrow & \oplus_{i=1}^n \mathbb{Z}[\pi]e_{c_i} & \xrightarrow{\partial} & \mathbb{Z}[\pi] & \longrightarrow & \mathbb{Z} \longrightarrow 0, \end{array}$$

where $(*)$ is the natural inclusion and $(**)$ sends e_d to the sum $\sum_{j=1}^p \text{sgn}(d_j) \left(\prod_{i=1}^{j-1} d_i \right) e_{d_j}$, where $\text{sgn}(d_j) = \pm 1$ depending on whether d_j is one of the c_i 's or one of the c_i^{-1} 's.

Combining this formula with the observation that $(1 - \rho_i)(\lambda) = (\alpha_i, \lambda)\alpha_i^\vee$, we see that the kernel of (21) is precisely $\ker \left[\sum_{i=1}^b \prod_{k=1}^{i-1} \rho_k : \oplus_{i=1}^b \mathbb{Z}\varepsilon_i \alpha_i^\vee \rightarrow \Lambda \right]$. \square

In particular, for the torsion subgroups of $H^1(C, j_*A)$ and $H^1(U, A)_{\text{tor}}$ we get

Corollary 4.7

$$H^1(C, j_*A)_{\text{tor}} = \left(\frac{\oplus_{i=1}^b \mathbb{Z}\varepsilon_i \alpha_i^\vee}{(1 - \rho_1, \dots, 1 - \rho_b)\Lambda} \right)_{\text{tor}}$$

$$H^1(U, A)_{\text{tor}} = H^1(U - \{s_0\}, A)_{\text{tor}} = \left(\frac{\Lambda^b}{(1 - \rho_1, \dots, 1 - \rho_b)\Lambda} \right)_{\text{tor}}.$$

Proof. Since $H^1(C, \Lambda)$ is torsion free, Proposition 4.6 implies that

$$H^1(C, j_*A)_{\text{tor}} = \left(\frac{\ker \left[\sum_{i=1}^b \prod_{k=1}^{i-1} \rho_k : \oplus_{i=1}^b \mathbb{Z}\varepsilon_i \alpha_i^\vee \rightarrow \Lambda \right]}{(1 - \rho_1, \dots, 1 - \rho_b)\Lambda} \right)_{\text{tor}}.$$

The corollary now follows by noticing that the saturation of $(1 - \rho_1, \dots, 1 - \rho_b)\Lambda$ inside the lattice $\oplus_{i=1}^b \mathbb{Z}\varepsilon_i \alpha_i^\vee$ is the same as the saturation of $(1 - \rho_1, \dots, 1 - \rho_b)\Lambda$ inside the lattice

$\ker \left[\sum_{i=1}^b \prod_{k=0}^{i-1} \rho_k : \oplus_{i=1}^b \mathbb{Z} \varepsilon_i \alpha_i^\vee \rightarrow \Lambda \right]$, since the latter lattice is a kernel to a map to Λ which is torsion free.

Finally, $H^1(U, A)_{\text{tor}} = H^1(U - \{s_0\}, A)_{\text{tor}}$ since the deletion of s_0 adds a copy of Λ as a direct summand. \square

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